Computational Fluid and Solid Mechanics

E. N. Dvorkin · M. B. Goldschmit

Nonlinear Continua

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Nonlinear Continua

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Library of Congress Control Number: 2005929275

ISBN-10 3-540-24985-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-24985-6 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media
springeronline.com

Typesetting: Dataconversion by authors
Final processing by PTP-Berlin Protago-TEX-Production GmbH, Germany
Cover-Design: deblik, Berlin
Printed on acid-free paper 62/3141/Yu – 5 4 3 2 1 0

To the Argentine system of public education

Preface

This book develops a modern presentation of Continuum Mechanics, oriented towards numerical applications in the fields of nonlinear analysis of solids, structures and fluids.

Kinematics of the continuum deformation, including pull-back/push-forward transformations between different configurations; stress and strain measures; objective stress rate and strain rate measures; balance principles; constitutive relations, with emphasis on elasto-plasticity of metals and variational principles are developed using general curvilinear coordinates.

Being tensor analysis the indispensable tool for the development of the continuum theory in general coordinates, in the appendix an overview of tensor analysis is also presented.

Embedded in the theoretical presentation, application examples are developed to deepen the understanding of the discussed concepts.

Even though the mathematical presentation of the different topics is quite rigorous; an effort is made to link formal developments with engineering physical intuition.

This book is based on two graduate courses that the authors teach at the Engineering School of the University of Buenos Aires and it is intended for graduate engineering students majoring in mechanics and for researchers in the fields of applied mechanics and numerical methods.

I am grateful to Klaus-Jürgen Bathe for introducing me to Computational Mechanics, for his enthusiasm, for his encouragement to undertake challenges and for his friendship.

I am also grateful to my colleagues, to my past and present students at the University of Buenos Aires and to my past and present research assistants at the Center for Industrial Research of FUDETEC because I have always learnt from them.

I want to thank Dr. Manuel Sadosky for inspiring many generations of Argentine scientists.

I am very grateful to my late father Israel and to my mother Raquel for their efforts and support.

Last but not least I want to thank my dear daughters Cora and Julia, my wife Elena and my friends (the best) for their continuous support.

Eduardo N. Dvorkin

I would like to thank Professors Eduardo Dvorkin and Sergio Idelsohn for introducing me to Computational Mechanics. I am also grateful to my students at the University of Buenos Aires and to my research assistants at the Center for Industrial Research of FUDETEC for their willingness and effort.

I want to recognize the permanent support of my mother Esther, of my sister Mónica and of my friends and colleagues.

Marcela B. Goldschmit

Contents

1

Introduction

The quantitative description of the deformation of continuum bodies, either solids or fluids subjected to mechanical and thermal loadings, is a challenging scientific field with very relevant technological applications.

1.1 Quantification of physical phenomena

The quantification of a physical phenomenon is performed through four different consecutive steps:

1. Observation of the physical phenomenon under study. Identification of its most relevant variables.
2. Formulation of a mathematical model that describes, in the framework of the assumptions derived from the previous step, the physical phenomenon.
3. Formulation of the numerical model that solves, within the required accuracy, the above-formulated mathematical model.
4. Assessment of the adequacy of the numerical results to describe the phenomenon under study.

1.1.1 Observation of physical phenomena

This is a crucial step that conditions the next three. Making an educated observation of a physical phenomenon means establishing a set of concepts and relations that will govern the further development of the mathematical model.

At this stage we also need to decide on the quantitative output that we shall require from the model.

1.1.2 Mathematical model

Considering the assumptions derived from the previous step and our knowledge on the physics of the phenomenon under study, we can establish the mathematical model that simulates it. This mathematical model, at least for the cases that fall within the field that this book intends to cover, is normally a system of partial differential equations (PDE) with established boundary and initial conditions.

1.1.3 Numerical model

Usually the PDE system that constitutes the mathematical model cannot be solved in closed form and the analyst needs to resort to a numerical model in order to arrive at the actual quantification of the phenomenon under study.

1.1.4 Assessment of the numerical results

The analyst has to judge if the numerical results are acceptable. This is a very important step and it involves:

- *Verification of the mathematical model*, that is to say, checking that the numerical results do not contradict any of the assumptions introduced for the formulation of the mathematical model and verification that the numerical results "make sense" by comparing them with the results of a "back-of-an-envelope" calculation (here, of course, we only compare orders of magnitude).
- *Verification of the numerical model*, the analyst has to assess if the numerical model can assure convergence to the unknown exact solution of the mathematical model when the numerical degrees of freedom are increased. The analyst must also check the stability of the numerical results when small perturbations are introduced in the data. If the results are not stable the analyst has to assess if the unstable numerical results represent an unstable physical phenomenon or if they are the result of an unacceptable numerical deficiency.
- *Validation of the mathematical/numerical model* comparing its predictions with experimental observations.

1.2 Linear and nonlinear mathematical models

When deriving the PDE system that constitutes the mathematical model of a physical phenomenon there are normally a number of nonlinear terms that appear in those equations. Considering always all the nonlinear terms, even if

their influence is negligible on the final numerical results, is mathematically correct; however, it may not be always practical.

The scientist or engineer facing the development of the mathematical model of a physical phenomenon has to decide which nonlinearities have to be kept in the model and which ones can be neglected. This is the main contribution of an analyst: formulating a model that is as simple as possible while keeping all the relevant aspects of the problem under analysis (Bathe 1996).

In many problems it is not possible to neglect all nonlinearities because the main features of the phenomenon under study lie in their consideration (Hodge, Bathe & Dvorkin 1986); in these cases the analyst must have enough physical insight into the problem so as to incorporate all the fundamental nonlinear aspects but *only* the fundamental ones. The more nonlinearities are introduced in the mathematical model, the more computational resources will be necessary to solve the numerical model and in many cases it may happen that the necessary computational resources are much larger than the available ones, making the analysis impossible.

Example 1.1. ◀◀◀◀◀

In the analysis of a solid under mechanical and thermal loads some of the nonlinearities that we may encounter when formulating the mathematical model are:

- Geometrical nonlinearities: they are introduced by the fact that the equilibrium equations have to be satisfied in the unknown deformed configuration of the solid rather than in the known unloaded configuration. When the analyst expects that for her/his practical purposes the difference between the deformed and unloaded configurations is negligible she/he may neglect this source of nonlinearity obtaining an important simplification in the mathematical model. An intermediate step would be to consider the equilibrium in the deformed configuration but to assume that the strains are very small (infinitesimal strains assumption). This also produces an important simplification in the mathematical model. Of course, all the simplifications introduced in the mathematical model have to be checked for their properness when examining the obtained numerical results.
- Contact-type boundary conditions: these are unilateral constraints in which the contact loads are distributed over an area that is a priori unknown to the analyst.
- Material nonlinearities: elastoplastic materials (e.g. metals); creep behavior of metals in high-temperature environments; nonlinear elastic materials (e.g. polymers); fracturing materials (e.g. concrete); etc.

◀◀◀◀◀

Example 1.2. ◀◀◀◀◀

In the analysis of a fluid flow under mechanical and thermal loads some of the nonlinearities that we may encounter when formulating the mathematical model are:

- Non-constant viscosity/compressibility (e.g. rheological materials and turbulent flows modeled using turbulence models).
- Convective acceleration terms for flows with Re>0 when the mathematical model is developed using an Eulerian formulation, which is the standard case.

◀◀◀◀◀

Example 1.3. ◀◀◀◀◀

In the analysis of a heat transfer problem some of the nonlinearities that we may encounter when formulating the mathematical model are:

- Temperature dependent thermal properties (e.g. phase changes).
- Radiation boundary conditions

◀◀◀◀◀

There are mathematical models in which the effects (outputs) are proportional to the causes (inputs); these are linear models. Examples of linear models are:

- linear elasticity problems,
- constant viscosity creeping flows,
- heat transfer problems in materials in which constant thermal properties are assumed and radiative boundary conditions are not considered,
- etc.

Deciding that the model that simulates a physical phenomenon is going to be linear is an analyst decision, after first considering and afterwards carefully neglecting, in the formulation of the mathematical model, all the sources of nonlinearity.

1.3 The aims of this book

This book intends to provide a modern and rigorous exposition of nonlinear continuum mechanics and even though it does not deal with computational implementations it is intended to provide the basis for them.

In the second chapter of the book we present a consistent description of the kinematics of the continuous media. In that chapter we introduce the concepts of pull-back, push-forward and Lie derivative requiring only from the reader

a previous knowledge of tensor analysis. Objective and covariant strain and strain rate measures are derived.

In the third chapter we discuss different stress measures that are energy conjugate to the strain rate measures presented in the previous chapter. Objective stress rate measures are derived.

In the fourth chapter we present the Reynolds transport theorem and then we use it to develop Eulerian and Lagrangian formulations for expressing the balance (conservation) of mass, momentum, moment of momentum and energy.

In the fifth chapter we develop an extensive presentation of constitutive relations for solids and fluids, with special focus on the elastoplasticity of metals.

Finally, in the sixth chapter we develop the variational approach to continuum mechanics, centering our presentation on the principle of virtual work and discussing also the principle of stationary potential energy and the Veubeke-Hu-Washizu variational principles.

The basic mathematical tool in the book is tensor calculus; in order to assure a common basis for all the readers, in the Appendix we present a review of this topic.

1.4 Notation

Throughout the book we shall use the summation convention; that is to say, in a Cartesian coordinate system

$$\begin{aligned} a_\alpha\ b_\alpha &= \sum_{\alpha=1}^{3} a_\alpha\ b_\alpha \\ a_{\alpha\beta}\ b_\beta &= \sum_{\beta=1}^{3} a_{\alpha\beta}\ b_\beta \qquad for \quad \alpha = 1,2,3\ , \end{aligned}$$

and in a general curvilinear system

$$\begin{aligned} a_i\ b^i &= \sum_{i=1}^{3} a_i\ b^i \\ a^{ij}\ b_j &= \sum_{j=1}^{3} a^{ij}\ b_j \qquad for \quad i = 1,2,3\ . \end{aligned}$$

Also, our notation is compatible with the notation introduced in continuum mechanics by Bathe (Bathe 1996). We shall define all notation at the point where we incorporate it.

2

Kinematics of the continuous media

In this chapter we are going to present a kinematic description of the deformation of continuous media. That is to say, we are going to describe the deformation without considering the loads that cause it and without introducing into the analysis the behavior of the material.

Some reference books for this chapter are: (Truesdell & Noll 1965, Truesdell 1966, Malvern 1969, Marsden & Hughes 1983).

2.1 The continuous media and its configurations

Continuum mechanics is the branch of mechanics that studies the motion of solids, liquids and gases under the *hypothesis of continuous media*. This hypothesis is an idealization of matter that disregards its atomic or molecular structure.

A *continuous body* is an open subset of the three-dimensional Euclidean space $\left(\Re^3\right)$ (Oden 1979) [1]. Each element "χ" of that subset is called a *point* or a *material particle*. The region of the Euclidean space occupied by the particles χ of the continuous body $\mathcal{B}$ at time t is called the *configuration* corresponding to t.

Above, we use the notion of *time* in a very general sense: as a coordinate that is used to *enumerate a series of events*. An *instant* t is a particular value of the time coordinate.

We can establish a bijective mapping (Oden 1979) between each point of space occupied by a material particle χ at t and an arbitrary curvilinear coordinate system $\{{}^t x^a,\ a = 1, 2, 3\}$.

The fact that at each instant t the set $\{{}^t x^a\}$ defines one and only one particle χ implies that in a continuum medium, different material particles

[1] The requirement of an open subset is introduced in order to eliminate the possible consideration of isolated points, sets in $\Re^3$ with zero volume, etc.

cannot occupy the same space location and that a material particle cannot be subdivided:

$$ {}^t x^a \;=\; {}^t x^a(\chi, t) \qquad ; \qquad \chi \;=\; \chi({}^t x^a)\ . \tag{2.1} $$

We assume that two arbitrary coordinate systems defined for the configuration at time t are related by continuous and differentiable functions:

$$ {}^t \tilde{x}^b \;=\; {}^t \tilde{x}^b({}^t x^a) \qquad ; \qquad {}^t x^a \;=\; {}^t x^a({}^t \tilde{x}^b)\ . \tag{2.2} $$

In a formal way, we say that the *configuration* of the body $\mathcal{B}$ corresponding to time t is an *homeomorphism* of $\mathcal{B}$ onto a region of the three dimensional Euclidean space ($\Re^3$) (Truesdell & Noll 1965). An homeomorphism (Oden 1979) is a bijective and continuous mapping with its inverse mapping also continuous.

The coordinates $\{{}^t x^a\}$ are called the *spatial coordinates* of the material particle χ in the configuration at time t.

We call the *motion* of the body $\mathcal{B}$ the evolution *from* a configuration at an instant t_1 *to* a configuration at an instant t_2.

We select *any* configuration of the body $\mathcal{B}$ as the *reference configuration* (e.g. the undeformed configuration, but not necessarily this one); also we can set the time origin so that in the reference configuration $t = 0$. In the reference configuration we define an arbitrary curvilinear coordinate system $\{{}^\circ x^A, A = 1, 2, 3\}$: the *material coordinates*. For the reference configuration Eqs.(2.1) are

$$ {}^\circ x^A \;=\; {}^\circ x^A(\chi) \qquad ; \qquad \chi \;=\; \chi({}^\circ x^A)\ . \tag{2.3} $$

From Eqs.(2.1) and (2.3) we obtain the bijective mapping ${}^t\phi$ between the configuration at time t and the reference configuration,

$$ {}^t x^a \;=\; {}^t \phi^a({}^\circ x^A, t) \qquad ; \qquad {}^\circ x^A \;=\; \left[{}^t \phi^{-1}\right]^A ({}^t x^a)\ . \tag{2.4} $$

In a *regular motion* the inverse mapping ${}^t\phi^{-1}$ exists and if ${}^t\phi \in C^r$ also ${}^t\phi^{-1} \in C^r$ (Marsden & Hughes 1983), where C^r is the set of all functions with continuous derivatives up to the order "r". The formal concept of regular motion agrees with the intuitive concept of a motion without material interpenetration.

From Eqs. (2.2) and (2.4) we get

$$ {}^t \tilde{x}^a \;=\; {}^t \tilde{x}^a({}^t x^b) \;=\; {}^t \tilde{x}^a \left[{}^t \phi^b({}^\circ x^A)\right] \;=\; {}^t \tilde{\phi}^a({}^\circ x^A)\ . \tag{2.5} $$

The mapping ${}^t\phi$ is a function of:

- the reference configuration,
- the configuration at t,
- the material coordinate system,
- the spatial coordinate system.

Since we restrict our presentation to the Euclidean space $\Re^3$, we can consider ${}^t\underline{\phi}$ as a vector. Hence, in Sect. 2.3.1 we are going to define the position and displacement vectors.

2.2 Mass of the continuous media

The continuous media have a non-negative scalar property named *mass*.

Our knowledge of Newton laws makes us relate the mass of a body with a measure of its inertia.

After (Truesdell 1966) we are going to assume a continuous mass distribution in the body $\mathcal{B}$. Concentrated masses do not belong to the field of Continuum Mechanics (therefore, Rational Mechanics is not part of Continuum Mechanics).

We define *the density* " ${}^t\rho$ " corresponding to the configuration at time t as

$$m \;=\; \int_{{}^tV} {}^t\rho \; {}^t\mathrm{d}V \;, \tag{2.6}$$

where, m: mass of body $\mathcal{B}$, ${}^tV$: volume of $\mathcal{B}$ in the configuration at time t.

Equation (2.6) incorporates an important postulate of Newtonian mechanics: *the mass of a body is constant in time.*

2.3 Motion of continuous bodies

2.3.1 Displacements

In the mapping ${}^t\underline{\boldsymbol{\phi}}$ $(\Re^3 \longrightarrow \Re^3)$ schematized in Fig. 2.1, at a given point (particle) χ, the vectors ${}^\circ\underline{\mathbf{g}}_A$ are the covariant base vectors[1] (Green & Zerna 1968) of the material coordinates $\{{}^\circ x^A\}$ (reference configuration; $t = 0$) and the vectors ${}^t\underline{\mathbf{g}}_a$ are the covariant base vectors of the spatial coordinates $\{{}^t x^a\}$ (spatial configuration corresponding to time t).

In the 3D Euclidean space we also define a fixed Cartesian system $\{{}^\circ z^\alpha \;\equiv\; {}^t z^\alpha \;\; \alpha = 1, 2, 3\}$ with a set of orthonormal base vectors $\underline{\mathbf{e}}_{\boldsymbol{\alpha}}$.

For the Cartesian coordinates of a particle χ in the reference configuration we use the triad $\{{}^\circ z^\alpha\}$ and for the Cartesian coordinates of the same particle in the spatial configuration we use the triad $\{{}^t z^\alpha\}$.

In the Cartesian system, the position vector ${}^\circ\underline{\mathbf{x}}$ of a particle χ in the reference configuration is

$${}^\circ\underline{\mathbf{x}}(\chi) \;=\; {}^\circ z^\alpha(\chi)\, \underline{\mathbf{e}}_{\boldsymbol{\alpha}} \;, \tag{2.7}$$

and the position vector ${}^t\underline{\mathbf{x}}$ of the particle χ in the spatial configuration is

$${}^t\underline{\mathbf{x}}(\chi, t) \;=\; {}^t z^\alpha(\chi, t)\, \underline{\mathbf{e}}_\alpha \;. \tag{2.8}$$

The *displacement vector* of the particle χ from the reference configuration to the spatial configuration is,

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[1] See Appendix.

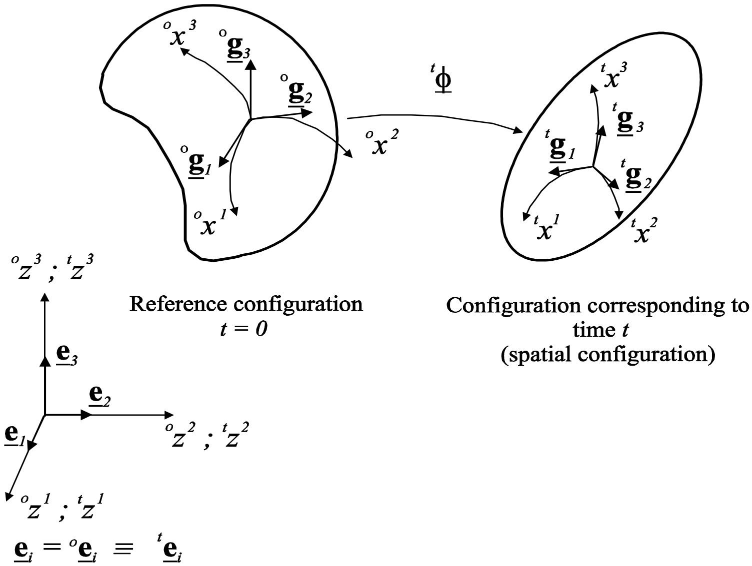


Fig. 2.1. Motion of continuous body

$$ {}^{t}\underline{\mathbf{u}}(\chi, t) \;=\; {}^{t}\underline{\mathbf{x}}(\chi, t) \;-\; {}^{\circ}\underline{\mathbf{x}}(\chi) \tag{2.9a}$$

and the Cartesian components of this vector are,

$$ {}^{t}u^{\alpha}(\chi, t) \;=\; {}^{t}z^{\alpha}(\chi, t) \;-\; {}^{\circ}z^{\alpha}(\chi)\ . \tag{2.9b}$$

2.3.2 Velocities and accelerations

During the *motion* ${}^{t}\underline{\phi}$, the *material velocity* of a particle χ in the t-configuration is

$$ {}^{t}\underline{\mathbf{v}}(\chi, t) \;=\; \frac{\partial\, {}^{t}\underline{\mathbf{x}}(\chi, t)}{\partial t} \;=\; \frac{\partial\, {}^{t}\underline{\mathbf{u}}(\chi, t)}{\partial t} \tag{2.10}$$

assuming that the time derivatives in Eq. (2.10) exist.

The material velocity vector is defined in the spatial configuration (see Fig. 2.2).

We can have, alternatively, the following functional dependencies:

$$ {}^{t}\underline{\mathbf{v}} \;=\; {}^{t}\underline{\mathbf{v}}({}^{\circ}x^{A}, t)\ , \tag{2.11a}$$

$$ {}^{t}\underline{\mathbf{v}} \;=\; {}^{t}\underline{\mathbf{v}}({}^{t}x^{a}, t)\ . \tag{2.11b}$$

Equation (2.11a) corresponds to a *Lagrangian (material) description of motion*, while Eq. (2.11b) corresponds to a *Eulerian (spatial) description of*

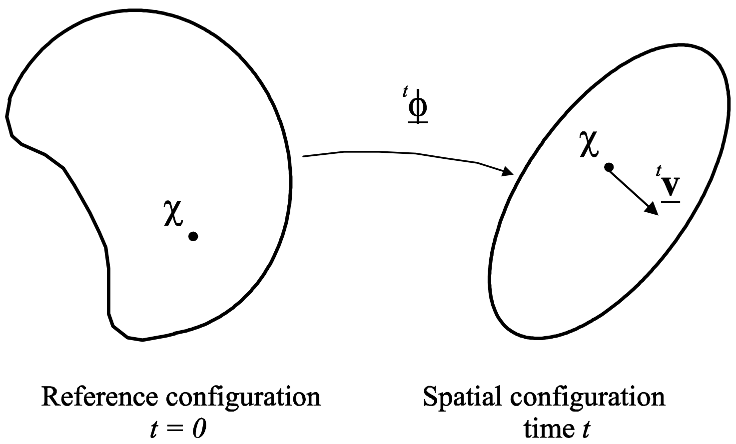


Fig. 2.2. Material velocity of a particle χ

motion. In general, the motion of solids is studied using Lagrangian descriptions while the motion of fluids is studied using Eulerian descriptions; however, this classification is by no means mandatory and combined descriptions have also been used in the literature (Belytschko, Lui & Moran 2000).

In any case, we can refer the velocity vector either to the spatial coordinates or to the fixed Cartesian coordinates,

$$ {}^t\underline{\mathbf{v}} = {}^t v^a \; {}^t\underline{\mathbf{g}}_a \,, \tag{2.12a} $$

$$ {}^t\underline{\mathbf{v}} = {}^t v^\alpha \; \underline{\mathbf{e}}_\alpha \,. \tag{2.12b} $$

Assuming an arbitrary tensor field ${}^t\boldsymbol{\eta} = {}^t\boldsymbol{\eta}(\chi, t) = {}^t\boldsymbol{\eta}(\,{}^\circ\underline{\mathbf{x}}, t)$ we define its *temporal material derivative* $(D^t\boldsymbol{\eta}/Dt)$ as the time rate of the tensor ${}^t\boldsymbol{\eta}$ when we keep constant the particle χ or, equivalently, when we keep constant the position vector ${}^\circ\underline{\mathbf{x}}$ in the reference configuration.

We call the *material acceleration* of a particle χ,

$$ {}^t\underline{\mathbf{a}} = \frac{D^t\underline{\mathbf{v}}}{Dt} \,. \tag{2.13} $$

In what follows, we determine the material acceleration vector considering the different combinations of:

$$ \text{description} \quad \left\{ \begin{array}{c} \textit{Lagrangian} \\ \\ \textit{Eulerian} \end{array} \right\} \quad + \quad \text{coordinates} \quad \left\{ \begin{array}{c} \textit{Spatial} \\ \\ \textit{Fixed Cartesian} \end{array} \right\} $$

- *Lagrangian description + spatial coordinates*

$$ {}^t\underline{\mathbf{a}} \;=\; \left[\; \frac{\partial\, {}^t v^a}{\partial t} \;+\; {}^t\Gamma^a_{bc}\;\; {}^t v^b\;\; {}^t v^c \right]\;\; {}^t\underline{\mathbf{g}}_a \;, \tag{2.14a}$$

where the ${}^t\Gamma^a_{bc}$ are the Christoffel symbols of the second kind of the spatial coordinates $\{{}^t x^a\}$ [2].

- *Lagrangian description + fixed Cartesian coordinates*

$$ {}^t\underline{\mathbf{a}} \;=\; \frac{\partial\, {}^t v^\alpha}{\partial t}\;\; \underline{\mathbf{e}}_{\boldsymbol{\alpha}} \;. \tag{2.14b}$$

- *Eulerian description + spatial coordinates*
  The material particle that at time t is at the spatial location $\{{}^t x^a\}$, at time $t + dt$ will be at $\{{}^t x^a + \; {}^t v^a\; \mathrm{d}t\}$. Hence,

$$ {}^{t+dt}\underline{\mathbf{v}} = \; {}^t\underline{\mathbf{v}} \;+\; \frac{\partial\, {}^t\underline{\mathbf{v}}}{\partial t}\,\mathrm{d}t + \frac{\partial\, {}^t\underline{\mathbf{v}}}{\partial\, {}^t x^b}\;\; {}^t v^b\; \mathrm{d}t $$

  Considering that the base vectors, in a general coordinate system, are functions of the position, we have

$$ {}^t\underline{\mathbf{a}} \;=\; \left[\frac{\partial\, {}^t v^a}{\partial t} \;+\; \frac{\partial\, {}^t v^a}{\partial\, {}^t x^b}\;\; {}^t v^b \;+\; {}^t\Gamma^a_{bc}\;\; {}^t v^b\;\; {}^t v^c \right]\;\; {}^t\underline{\mathbf{g}}_a \;. \tag{2.15a}$$

- *Eulerian description + fixed Cartesian coordinates*

$$ {}^t\underline{\mathbf{a}} \;=\; \left[\frac{\partial\, {}^t v^\alpha}{\partial t} \;+\; \frac{\partial\, {}^t v^\alpha}{\partial\, {}^t z^\beta}\;\; {}^t v^\beta \right]\;\; \underline{\mathbf{e}}_{\boldsymbol{\alpha}} \;. \tag{2.15b}$$

2.4 Material and spatial derivatives of a tensor field

Let ${}^t\boldsymbol{\eta}$ be an arbitrary tensor field, a function of time, and using a Lagrangian description of motion we get,

$$ {}^t\boldsymbol{\eta} \;=\; {}^t\boldsymbol{\eta}({}^\circ x^A, t) \;. \tag{2.16a}$$

Now, using an Eulerian description of motion, we get

$$ {}^t\boldsymbol{\eta} \;=\; {}^t\boldsymbol{\eta}({}^t x^a, t) \;. \tag{2.16b}$$

We indicate the first *material time derivative* (following the particle) as

$$ {}^t\dot{\boldsymbol{\eta}} = \frac{D\, {}^t\boldsymbol{\eta}}{Dt} \;=\; \frac{\partial\, {}^t\boldsymbol{\eta}({}^\circ x^A, t)}{\partial t}\, \Big|_{{}^\circ x_A} \;. \tag{2.17}$$

[2] See Appendix.

The first *spatial time derivative* of a tensor ${}^t\boldsymbol{\eta}$ defined using a Eulerian description is simply indicated as $\frac{\partial^t \boldsymbol{\eta}({}^t x^a, t)}{\partial t}\ |_{{}^t x^a}$.

Let us now calculate the material time derivative of a tensor ${}^t\boldsymbol{\eta}$ defined using an Eulerian description,

$$ {}^t\boldsymbol{\eta} \ = \ {}^t\eta^{a\cdots b}{}_{c\cdots d}({}^t x^a, t)\ {}^t\underline{\mathbf{g}}_a \cdots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \cdots\ {}^t\underline{\mathbf{g}}^d \tag{2.18}$$

it is easy to derive the following relation (Truesdell & Noll 1965, Slattery 1972),

$$ {}^t\dot{\boldsymbol{\eta}} \ = \ \frac{\partial^t \boldsymbol{\eta}}{\partial t} \ + \ {}^t v^p\ \frac{\partial^t \boldsymbol{\eta}}{\partial^t x^p}\ . \tag{2.19}$$

Using the spatial gradient of the tensor ${}^t\boldsymbol{\eta}$ [3]:

$$\begin{aligned} \underline{\boldsymbol{\nabla}}\ {}^t\boldsymbol{\eta} = {}&{}^t\underline{\mathbf{g}}^p\ \frac{\partial^t \boldsymbol{\eta}}{\partial^t x^p} \qquad\qquad (2.20\mathrm{a}) \\ = {}&[\frac{\partial^t \eta^{a\cdots b}{}_{c\cdots d}}{\partial^t x^p} + {}^t\eta^{k\cdots b}{}_{c\cdots d}\ {}^t\Gamma^a_{kp} + \ldots + {}^t\eta^{a\cdots k}{}_{c\cdots d}\ {}^t\Gamma^b_{kp} \\ &- {}^t\eta^{a\cdots b}{}_{k\cdots d}\ {}^t\Gamma^k_{pc} - \ldots - {}^t\eta^{a\cdots b}{}_{c\cdots k}\ {}^t\Gamma^k_{pd}\ ]\ \ {}^t\underline{\mathbf{g}}^p\ {}^t\underline{\mathbf{g}}_a \cdots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \cdots\ {}^t\underline{\mathbf{g}}^d \end{aligned}$$

we can rewrite Eq. (2.19) as,

$$ {}^t\dot{\boldsymbol{\eta}} \ = \ \frac{\partial^t \boldsymbol{\eta}}{\partial t} \ + \ \ {}^t\underline{\mathbf{v}} \cdot (\underline{\boldsymbol{\nabla}}{}^t\boldsymbol{\eta})\ . \tag{2.20b}$$

2.5 Convected coordinates

Let us consider a body $\mathcal{B}$ and define in its reference configuration a system of curvilinear coordinates $\{\theta^i, i = 1,2,3\}$.

The curvilinear coordinate system $\{\theta^i\}$ is a *convected coordinate system* (Flügge 1972) if, when the body $\mathcal{B}$ undergoes a deformation process, for each configuration, the triad $\{\theta^i\}$ that defines a particle χ, is the same as in the reference configuration.

2.6 The deformation gradient tensor

Let us consider the motion of the body $\mathcal{B}$ represented in Fig. 2.1. For the reference configuration ($t = 0$) we can write at the point (particle) χ:

$$ {}^\circ\mathbf{d}\underline{\mathbf{x}} \ = \ {}^\circ\mathrm{d}x^A\ {}^\circ\underline{\mathbf{g}}_A \tag{2.21a}$$

[3] See Appendix.

where the vector ${}^{\circ}\mathrm{d}\underline{\mathbf{x}}$ at the point χ in the reference configuration is called a *material line element or fiber* (Ogden 1984).

Due to the motion ${}^{t}\phi$, the above defined fiber is transformed into a fibre in the spatial configuration,

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}} \;=\; {}^{t}\mathrm{d}x^{a}\;{}^{t}\underline{\mathbf{g}}_{a} \;. \tag{2.21b} $$

We now define a second-order tensor: ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}$, the *deformation gradient tensor* at χ,

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}} \;. \tag{2.22} $$

From the above equation,

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \;=\; \frac{\partial\, {}^{t}x^{a}}{\partial\, {}^{\circ}x^{A}} \;\; {}^{t}\underline{\mathbf{g}}_{a} \;\; {}^{\circ}\underline{\mathbf{g}}^{A} \;. \tag{2.23} $$

Using the first of Eqs. (2.4) we get the following functional relation,

$$ {}^{t}_{\circ}X^{a}{}_{A} \;=\; {}^{t}_{\circ}X^{a}{}_{A}({}^{\circ}x^{B}, t) \;. \tag{2.24} $$

From Eq.(2.23), we see that the tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}$ has one base vector in the reference configuration and the other in the spatial configuration. Hence, it is a *two-point tensor*, (Marsden & Hughes 1983, Lubliner 1985).

It is important to note that ${}^{t}_{\circ}X^{a}{}_{A}$ is a function of,

- the motion of the body (${}^{t}\phi$),
- the material coordinate system,
- the spatial coordinate system.

For a regular motion, the tensor that is the inverse of ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}$ at ${}^{t}\underline{\mathbf{x}}$ is,

$$ {}^{\circ}\mathrm{d}\underline{\mathbf{x}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1} \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}} \tag{2.25} $$

where,

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1} \;=\; \frac{\partial\, {}^{\circ}x^{A}}{\partial\, {}^{t}x^{a}} \;\; {}^{\circ}\underline{\mathbf{g}}_{A} \;\; {}^{t}\underline{\mathbf{g}}^{a} \;. \tag{2.26} $$

Using the second of Eqs. (2.4), we get the following functional relation,

$$ ({}^{t}_{\circ}X^{-1})^{A}{}_{a} \;=\; ({}^{t}_{\circ}X^{-1})^{A}{}_{a}\;({}^{t}x^{b}, t) \;. \tag{2.27} $$

We define the *transpose* of ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}$ at χ using the following relation (Marsden & Hughes 1983, Strang 1980),

$$ ({}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}) \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}} \;=\; {}^{\circ}\mathrm{d}\underline{\mathbf{x}} \;\cdot\; ({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}) \;, \tag{2.28a} $$

if we now define ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \;=\; ({}^{t}_{\circ}X^{T})^{A}{}_{a} \;\; {}^{\circ}\underline{\mathbf{g}}_{A} \;\; {}^{t}\underline{\mathbf{g}}^{a}$, we get from Eq.(2.28$a$),

$$ {}^t_\circ X^a{}_A \ {}^\circ\mathrm{d}x^A \ {}^t\mathrm{d}^\circ x^b \ {}^t g_{ab} = {}^\circ\mathrm{d}x^A \ {}^\circ g_{AB} \ ({}^t_\circ X^T)^B_b \ {}^t\mathrm{d}x^b \tag{2.28b} $$

hence,

$$ ({}^t_\circ X^T)^B_b = {}^t_\circ X^a{}_A \ {}^t g_{ab} \ {}^\circ g^{BA} \ . \tag{2.28c} $$

and therefore,

$$ ({}^t_\circ X^T)^B_b = {}^t_\circ X_b{}^B \ . \tag{2.28d} $$

In the above equations, ${}^t g_{ab} = {}^t\underline{\mathbf{g}}_a \cdot {}^t\underline{\mathbf{g}}_b$ are the covariant components of the metric tensor in the spatial configuration at $\{{}^t x^A\}$ and ${}^\circ g^{AB} = {}^\circ\underline{\mathbf{g}}^A \cdot {}^\circ\underline{\mathbf{g}}^B$ are the contravariant components of the metric tensor in the reference configuration at $\{{}^\circ x^a\}$ [4].

Referring the body $\mathcal{B}$ to a fixed Cartesian system, and using Eqs. (2.9a-2.9b),

$$ {}^t_\circ X_{\alpha\beta} = \delta_{\alpha\beta} + \frac{\partial {}^t u^\alpha}{\partial {}^\circ z^\beta} \ , \tag{2.29a} $$

$$ ({}^t_\circ X^{-1})_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\partial {}^t u^\alpha}{\partial {}^t z^\beta} \ , \tag{2.29b} $$

$$ ({}^t_\circ X^T)_{\alpha\beta} = {}^t_\circ X_{\beta\alpha} \ , \tag{2.29c} $$

where

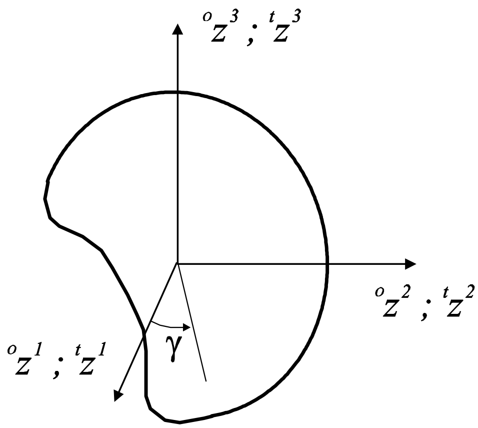
$\delta_{\alpha\beta}$: components of the Kronecker-delta.

It is important to remember that when a problem is referred to a Cartesian system we do not need to make the distinction between covariant and contravariant tensorial components (Green & Zerna 1968).

Example 2.1. ◀◀◀◀◀
In a fixed Cartesian system, a rigid translation is represented by a deformation gradient tensor with components ${}^t_\circ X_{\alpha\beta} = \delta_{\alpha\beta}$. ◀◀◀◀◀

Example 2.2. ◀◀◀◀◀
In the rigid rotation represented in the following figure,

[4] See Appendix.



Rigid rotation

We can directly calculate the components ${}^t_\circ X_{\alpha\beta}$ using Eq. (2.29a) but it may be simpler to consider the following sequence:

Step 1: Transformation from Cartesian coordinates to cylindrical coordinates in the reference configuration.

$$\begin{aligned} {}^\circ\theta^1 &= \sqrt{({}^\circ z^1)^2 + ({}^\circ z^2)^2} \\ {}^\circ\theta^2 &= \tan^{-1}\left(\frac{{}^\circ z^2}{{}^\circ z^1}\right) \qquad ; \qquad (X_1)^l{}_\beta = \frac{\partial^\circ\theta^l}{\partial^\circ z^\beta} \\ {}^\circ\theta^3 &= {}^\circ z^3 \ . \end{aligned}$$

Without any motion, the change of the coordinate system in the reference configuration produces a deformation gradient tensor.

Step 2: Rigid rotation.

$$\begin{aligned} {}^t\theta^1 &= {}^\circ\theta^1 \\ {}^t\theta^2 &= {}^\circ\theta^2 + \gamma \qquad ; \qquad (X_2)^p{}_l = \frac{\partial^t\theta^p}{\partial^\circ\theta^l} \\ {}^t\theta^3 &= {}^\circ\theta^3 \ . \end{aligned}$$

Step 3: Transformation from cylindrical coordinates to Cartesian coordinates in the spatial configuration.

$$\begin{aligned} {}^t z^1 &= {}^t\theta^1 cos\ {}^t\theta^2 \\ {}^t z^2 &= {}^t\theta^1 \sin\ {}^t\theta^2 \qquad ; \qquad (X_3)^\alpha{}_p = \frac{\partial^t z^\alpha}{\partial^t\theta^p} \\ {}^t z^3 &= {}^t\theta^3 \ . \end{aligned}$$

Without any further motion, the change of the coordinate system in the spatial configuration produces a deformation gradient tensor.

Using the chain rule,

$$
\begin{aligned}
{}^t_\circ X_{\alpha\beta} &= \frac{\partial^t z^\alpha}{\partial^\circ z^\beta} = \frac{\partial^t z^\alpha}{\partial^t \theta^p}\ \frac{\partial^t \theta^p}{\partial^\circ \theta^l}\ \frac{\partial^\circ \theta^l}{\partial^\circ z^\beta} \\
{}^t_\circ X_{\alpha\beta} &= (X_3)^\alpha{}_p\ \ (X_2)^p{}_l\ \ (X_1)^l{}_\beta
\end{aligned}
$$

For **step 1**, the derivation of $(X_1^{-1})^\beta{}_l$ can be done by inspection. The array of these components is,

$$
\left[X_1^{-1}\right] = \begin{bmatrix} \cos\ {}^\circ\theta^2 & -\ {}^\circ\theta^1 \sin\ {}^\circ\theta^2 & 0 \\ \sin\ {}^\circ\theta^2 & {}^\circ\theta^1 \cos\ {}^\circ\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,
$$

inverting

$$
[X_1] = \begin{bmatrix} \cos\ {}^\circ\theta^2 & \sin\ {}^\circ\theta^2 & 0 \\ -\frac{1}{{}^\circ\theta^1}\ \sin\ {}^\circ\theta^2 & \frac{1}{{}^\circ\theta^1} \cos\ {}^\circ\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} .
$$

For **step 2**,

$$
[X_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .
$$

For **step 3**,

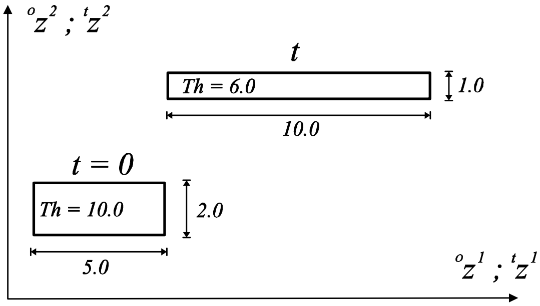
$$
[X_3] = \begin{bmatrix} \cos\ {}^t\theta^2 & -\ {}^t\theta^1 \sin\ {}^t\theta^2 & 0 \\ \sin\ {}^t\theta^2 & {}^t\theta^1 \cos\ {}^t\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} .
$$

Finally,

$$
\left[{}^t_\circ X\right] = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} .
$$

An important feature of the above matrix is that $[{}^t_\circ X]^T\ \ [{}^t_\circ X]\ = \ [I]$, where $[I]$ is the unit matrix. That is to say, the matrix $[{}^t_\circ X]$ is *orthogonal.*◀◀◀◀◀

Example 2.3. ____◀◀◀◀◀
For the motion represented in the following figure, we can derive the deformation gradient tensor directly by inspection,

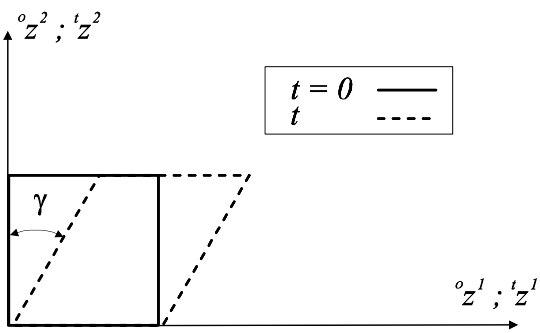


Simple deformation process

$$\left[{}^{t}_{\circ}X\right] = \begin{bmatrix} 2.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.6 \end{bmatrix} .$$

Example 2.4.

For the motion represented in the following figure,



Shear deformation

$$\begin{aligned} {}^{t}z_1 &= {}^{\circ}z^1 + {}^{\circ}z^2 \;\tan\gamma \\ {}^{t}z^2 &= {}^{\circ}z^2 \\ {}^{t}z^3 &= {}^{\circ}z^3 \;, \end{aligned}$$

therefore,

$$\left[{}^{t}_{\circ}X\right] = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

When there is a *sequence of motions* (some of them can be just a change of coordinate system) like the sequence depicted in Fig. 2.3, we can generalize the result in Example 2.2,

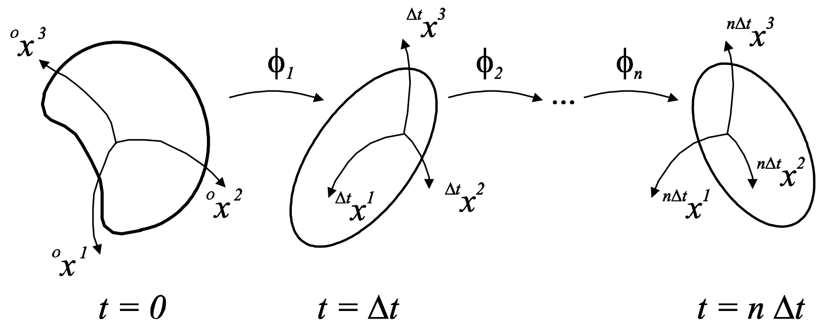


Fig. 2.3. Sequence of motions

$$ {}^{n\Delta t}_{\circ}X^{a}{}_{P} = \frac{\partial^{n\Delta t}x^{a}}{\partial^{\circ}x^{P}} = \frac{\partial^{n\Delta t}x^{a}}{\partial^{(n-1)\Delta t}x^{b}} \cdots\cdots \frac{\partial^{\Delta t}x^{l}}{\partial^{\circ}x^{P}} \ . \tag{2.30a}$$

Therefore,

$$ {}^{n\Delta t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{n\Delta t}_{(n-1)\Delta t}\underline{\underline{\mathbf{X}}} \cdot {}^{(n-1)\Delta t}_{(n-2)\Delta t}\underline{\underline{\mathbf{X}}} \cdots {}^{2\Delta t}_{\Delta t}\underline{\underline{\mathbf{X}}} \cdot {}^{\Delta t}_{\circ}\underline{\underline{\mathbf{X}}} \ . \tag{2.30b}$$

Example 2.5. ◀◀◀◀◀
It is easy to show that when using a convected coordinate system

$$ {}^{t}_{\circ}X^{a}_{\ b} = \delta^{a}_{b} $$

where the δ^{a}_{b} are the mixed components of the Kronecker delta tensor. ◀◀◀◀◀

For a motion ${}^{t}\phi$ we define at a point χ the *Jacobian of the transformation* (Truesdell & Noll 1965),

$$ {}^{t}J(\chi, t) = \frac{{}^{t}\mathrm{d}V}{{}^{\circ}\mathrm{d}V} \ , \tag{2.31}$$

where ${}^{\circ}\mathrm{d}V$ is a differential volume in the reference configuration and ${}^{t}\mathrm{d}V$ is the corresponding differential volume in the spatial configuration. Since in a regular motion, a nonzero volume in the reference configuration cannot be

collapsed into a point in the spatial configuration and vice versa (Aris 1962), ${}^{t}J$ and ${}^{t}J^{-1}$ cannot be zero.

In a fixed Cartesian system we can define, for a motion ${}^{t}\phi$,

$$ {}^{t}z^{\alpha} = {}^{t}z^{\alpha}({}^{\circ}z^{\beta},\ t)\ . \tag{2.32}$$

The vectors (Hildebrand 1976)

$$ {}^{t}\underline{\mathbf{t}}_{\beta} = \frac{\partial^{t} z^{\alpha}}{\partial^{\circ} z^{\beta}}\ {}^{t}\underline{\mathbf{e}}_{\alpha} \qquad (\beta = 1, 2, 3) \tag{2.33}$$

are the base vectors, in the spatial configuration, of a convected coordinate system $\{{}^{\circ}z^{\alpha}\}$ and

$$ {}^{\circ}\mathrm{d}V = {}^{\circ}\mathrm{d}z^{1}\ {}^{\circ}\mathrm{d}z^{2}\ {}^{\circ}\mathrm{d}z^{3} \tag{2.34a}$$

$$ {}^{t}\mathrm{d}V = {}^{\circ}\mathrm{d}z^{1}\ {}^{\circ}\mathrm{d}z^{2}\ {}^{\circ}\mathrm{d}z^{3}\ [\ {}^{t}\underline{\mathbf{t}}_{1} \cdot (\ {}^{t}\underline{\mathbf{t}}_{2} \times {}^{t}\underline{\mathbf{t}}_{3}\ )]\ . \tag{2.34b}$$

Therefore,

$$ {}^{t}\mathrm{d}V = \det \begin{bmatrix} \frac{\partial^{t} z^{1}}{\partial^{\circ} z^{1}} & \frac{\partial^{t} z^{2}}{\partial^{\circ} z^{1}} & \frac{\partial^{t} z^{3}}{\partial^{\circ} z^{1}} \\ \frac{\partial^{t} z^{1}}{\partial^{\circ} z^{2}} & \frac{\partial^{t} z^{2}}{\partial^{\circ} z^{2}} & \frac{\partial^{t} z^{3}}{\partial^{\circ} z^{2}} \\ \frac{\partial^{t} z^{1}}{\partial^{\circ} z^{3}} & \frac{\partial^{t} z^{2}}{\partial^{\circ} z^{3}} & \frac{\partial^{t} z^{3}}{\partial^{\circ} z^{3}} \end{bmatrix} {}^{\circ}\mathrm{d}V\ . \tag{2.34c}$$

Since transpose matrices have the same determinant,

$$ {}^{t}\mathrm{d}V = \left|{}^{t}_{\circ}X\right| {}^{\circ}\mathrm{d}V \tag{2.34d}$$

where $\left|{}^{t}_{\circ}X\right| = \det\left[{}^{t}_{\circ}X\right]$.

When the motion of a body $\mathcal{B}$ is referred to a fixed Cartesian coordinate system (Malvern 1969),

$$ {}^{t}J(\chi, t) = \left|{}^{t}_{\circ}X\right|\ . \tag{2.34e}$$

When in the reference configuration we use a curvilinear system $\{{}^{\circ}x^{A}\}$ and in the spatial configuration a system $\{{}^{t}x^{a}\}$, (Marsden & Hughes 1983):

$$ {}^{t}J = \det\left[\frac{\partial^{t} z^{\alpha}}{\partial^{\circ} z^{\beta}}\right] = \det\left[\frac{\partial^{t} z^{\alpha}}{\partial^{t} x^{a}}\ \frac{\partial^{t} x^{a}}{\partial^{\circ} x^{A}}\ \frac{\partial^{\circ} x^{A}}{\partial^{\circ} z^{\beta}}\right]\ . \tag{2.34f}$$

Following the reference (Green & Zerna 1968) and doing some algebra we can show that:

$$ \left|{}^{t}g_{ab}\right| = \det\left[{}^{t}g_{ab}\right] = \left[\det\left[\frac{\partial^{t} z^{\alpha}}{\partial^{t} x^{a}}\right]\right]^{2} \tag{2.34g}$$

and

$$ \left|\ {}^{\circ}g_{AB}\right| = \det\left[\ {}^{\circ}g_{AB}\right] = \left[\det\left[\frac{\partial^{\circ} x^{A}}{\partial^{\circ} z^{\beta}}\right]\right]^{-2}\ . \tag{2.34h}$$

Finally,

$$ {}^{t}J(\chi,t) = \left|{}^{t}_{\circ}X\right| \sqrt{\frac{|{}^{t}g_{ab}|}{|{}^{\circ}g_{AB}|}} \ . \tag{2.34i}$$

Example 2.6. ◀◀◀◀◀
In a isocoric deformation (without change of volume) ${}^{t}J = 1$, hence

$$\left|{}^{t}_{\circ}X\right| \sqrt{|{}^{t}g_{ab}|} = \sqrt{|{}^{\circ}g_{AB}|} \ .$$

○ When in the problem we use a fixed Cartesian coordinate system, we have

$$\left|{}^{t}_{\circ}X\right| = 1 \ .$$

○ When we use convective coordinates

$$\left|{}^{t}g_{ab}\right| = |{}^{\circ}g_{AB}| \ .$$

◀◀◀◀◀

2.7 The polar decomposition

The polar decomposition theorem (Truesdell & Noll 1965, Truesdell 1966, Malvern 1969, Marsden & Hughes 1983) is a fundamental step in the development of the kinematic description of continuous body motions. It allows us to locally (at a point χ) decompose any motion into a pure deformation motion followed by a pure rotation motion or vice versa.

2.7.1 The Green deformation tensor

The *Green deformation tensor* is defined at a point χ as

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \ . \tag{2.35}$$

In some references, e.g. (Truesdell & Noll 1965, Truesdell 1966), the above tensor is referred to as *right Cauchy-Green deformation tensor* .

Using Eq. (2.28c), we get

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} = \left[{}^{t}_{\circ}X^{a}{}_{A}\ {}^{t}g_{ab}\ {}^{\circ}g^{AB}\ {}^{\circ}\underline{\mathbf{g}}_{B}\ {}^{t}\underline{\mathbf{g}}^{b}\right] \cdot \left[{}^{t}_{\circ}X^{d}{}_{D}\ {}^{t}\underline{\mathbf{g}}_{d}\ {}^{\circ}\underline{\mathbf{g}}^{D}\right] , \tag{2.36a}$$

and therefore,

$$ {}^t_\circ\underline{\underline{\mathbf{C}}} = \left[{}^t_\circ X^a{}_A \; {}^t_\circ X^b{}_D \; {}^t g_{ab} \; {}^\circ g^{AB}\right] \; {}^\circ\underline{\mathbf{g}}_B \; {}^\circ\underline{\mathbf{g}}^D \, . \tag{2.36b} $$

It is important to note that the Green deformation tensor is *defined in the reference configuration*.

Using an equivalent definition of transposed tensor to that given in Eq.(2.28a), we can write

$$ ({}^t_\circ\underline{\underline{\mathbf{C}}} \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_1) \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_2 = {}^\circ\mathrm{d}\underline{\mathbf{x}}_1 \cdot ({}^t_\circ\underline{\underline{\mathbf{C}}}^{\mathbf{T}} \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_2) \tag{2.37a} $$

where ${}^\circ\mathrm{d}\underline{\mathbf{x}}_1$ and ${}^\circ\mathrm{d}\underline{\mathbf{x}}_2$ are two arbitrary vectors defined in the reference configuration at the point under study; after some algebra Eq. (2.37a) leads to

$$ {}^t_\circ C^A{}_B = ({}^t_\circ C^T)^A{}_B = {}^t_\circ C_B{}^A \, . \tag{2.37b} $$

The above equation indicates that the Green deformation tensor is *symmetric.*

For an arbitrary vector ${}^t\mathrm{d}\underline{\mathbf{x}}$ defined in the spatial configuration we can write the following equalities:

$$ {}^t\mathrm{d}\underline{\mathbf{x}} = {}^t\mathrm{d}x^a \; {}^t\underline{\mathbf{g}}_a = {}^t_\circ X^a{}_A \; {}^\circ\mathrm{d}x^A \; {}^t\underline{\mathbf{g}}_a \tag{2.38a} $$

$$ {}^t\mathrm{d}\underline{\mathbf{x}} = {}^t\mathrm{d}x_b \; {}^t\underline{\mathbf{g}}^b = {}^t_\circ X^d{}_B \; {}^\circ g^{BR} \; {}^\circ\mathrm{d}x_R \; {}^t g_{db} \; {}^t\underline{\mathbf{g}}^b \, . \tag{2.38b} $$

Hence,

$$ {}^t\mathrm{d}\underline{\mathbf{x}} \cdot {}^t\mathrm{d}\underline{\mathbf{x}} = {}^t_\circ X^a{}_A \; {}^t_\circ X^d{}_B \; {}^t g_{da} \; {}^\circ g^{BR} \; {}^\circ\mathrm{d}x^A \; {}^\circ\mathrm{d}x_R \, . \tag{2.38c} $$

Using Eq. (2.36b), we write

$$ {}^t\mathrm{d}\underline{\mathbf{x}} \cdot {}^t\mathrm{d}\underline{\mathbf{x}} = {}^t_\circ C^R{}_A \; {}^\circ\mathrm{d}x^A \; {}^\circ\mathrm{d}x_R = {}^\circ\mathrm{d}\underline{\mathbf{x}} \cdot {}^t_\circ\underline{\underline{\mathbf{C}}} \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}} \, . \tag{2.38d} $$

Considering that:

- ${}^t\mathrm{d}\underline{\mathbf{x}} \cdot {}^t\mathrm{d}\underline{\mathbf{x}} \geq 0$.
- ${}^t\mathrm{d}\underline{\mathbf{x}} \cdot {}^t\mathrm{d}\underline{\mathbf{x}} = 0 \iff |{}^t\mathrm{d}\underline{\mathbf{x}}| = 0$.
- If ${}^t\phi$ is a regular motion, $|{}^t\mathrm{d}\underline{\mathbf{x}}| = 0 \iff |{}^\circ\mathrm{d}\underline{\mathbf{x}}| = 0$ we conclude that ${}^t_\circ\underline{\underline{\mathbf{C}}}$ is a *positive-definite* tensor (Strang 1980).

2.7.2 The right polar decomposition

We define, in the reference configuration at the point under study, the *right stretch tensor* as

$$ {}^t_\circ\underline{\underline{\mathbf{U}}} = \left[{}^t_\circ\underline{\underline{\mathbf{C}}}\right]^{1/2} \tag{2.39} $$

and it follows immediately that the tensor ${}^t_\circ\underline{\underline{\mathbf{U}}}$ inherits from ${}^t_\circ\underline{\underline{\mathbf{C}}}$ the properties of *symmetry* and *positive-definiteness* (Malvern 1969).

We define the *right polar decomposition* as a *multiplicative decomposition* of the tensor ${}^t_\circ\underline{\underline{\mathbf{X}}}$ into a *symmetric* tensor $({}^t_\circ\underline{\underline{\mathbf{U}}})$ premultiplied by a tensor that we will show is *orthogonal*: the rotation tensor $({}^t_\circ\underline{\underline{\mathbf{R}}})$. Hence,

$$ {}^t_\circ\underline{\underline{\mathbf{X}}} = {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}} \, . \tag{2.40}$$

In order to show that ${}^t_\circ\underline{\underline{\mathbf{R}}}$ is an orthogonal tensor we have to show that:

$$\begin{aligned} (i) \quad {}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} &= {}^\circ\underline{\underline{\mathbf{g}}} \\ (ii) \quad {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}^T &= {}^t\underline{\underline{\mathbf{g}}} \end{aligned}$$

where ${}^\circ\underline{\underline{\mathbf{g}}} = \delta^A{}_B \, {}^\circ\underline{\mathbf{g}}_A \, {}^\circ\underline{\mathbf{g}}^B$ is the unit tensor of the reference configuration and ${}^t\underline{\underline{\mathbf{g}}} = \delta^a{}_b \, {}^t\underline{\mathbf{g}}_a \, {}^t\underline{\mathbf{g}}^b$ is the unit tensor of the spatial configuration ($\delta^A{}_B$ and $\delta^a{}_b$ are Kronecker deltas [5]).

To prove the first equality (i), we start from Eq. (2.40) and get

$$ {}^t_\circ\underline{\underline{\mathbf{R}}} = {}^t_\circ\underline{\underline{\mathbf{X}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}^{-1} = {}^t_\circ X^a{}_A \, ({}^t_\circ U^{-1})^A{}_B \, {}^t\underline{\mathbf{g}}_a \, {}^\circ\underline{\mathbf{g}}^B \qquad . \tag{2.41}$$

The tensor ${}^t_\circ\underline{\underline{\mathbf{R}}}$ defined by the above equation is, in the same sense as ${}^t_\circ\underline{\underline{\mathbf{X}}}$, a *two-point tensor*. The components of ${}^t_\circ\underline{\underline{\mathbf{R}}}$ are:

$$ {}^t_\circ R^a{}_A = {}^t_\circ X^a{}_L \, ({}^t_\circ U^{-1})^L{}_A \, , \tag{2.42a}$$

and using an similar equation to (2.28c) we have

$$ ({}^t_\circ R^T)^B_{\ b} = {}^t_\circ X^a{}_L \, ({}^t_\circ U^{-1})^L{}_A \, {}^t g_{ab} \, {}^\circ g^{BA} \, , \tag{2.42b}$$

hence,

$$ {}^t_\circ\underline{\underline{\mathbf{R}}}^T = {}^t_\circ X^a{}_L \, ({}^t_\circ U^{-1})^L{}_A \, {}^t g_{ab} \, {}^\circ g^{BA} \, {}^\circ\underline{\mathbf{g}}_B \, {}^t\underline{\mathbf{g}}^b \, . \tag{2.42c}$$

Considering that

$$ {}^t_\circ\underline{\underline{\mathbf{X}}}^T = {}^t_\circ X^d{}_R \, {}^t g_{db} \, {}^\circ g^{RB} \, {}^\circ\underline{\mathbf{g}}_B \, {}^t\underline{\mathbf{g}}^b \, , \tag{2.42d}$$

and

$$ {}^t_\circ\underline{\underline{\mathbf{U}}}^{-T} = ({}^t_\circ U^{-1})^{\ L}_{D} \, {}^\circ\underline{\mathbf{g}}^D \, {}^\circ\underline{\mathbf{g}}_L \, , \tag{2.42e}$$

$$ {}^t_\circ\underline{\underline{\mathbf{U}}}^{-T} \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^T = ({}^t_\circ U^{-1})^{\ L}_{D} \; {}^t_\circ X^d{}_R \; \delta^R_L \; {}^t g_{db} \; {}^\circ\underline{\mathbf{g}}^D \, {}^t\underline{\mathbf{g}}^b \, , \tag{2.42f}$$

and since ${}^\circ\underline{\mathbf{g}}^D = {}^\circ g^{DB} \, {}^\circ\underline{\mathbf{g}}_B$ we write

$$ {}^t_\circ\underline{\underline{\mathbf{U}}}^{-T} \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^T = ({}^t_\circ U^{-1})^{\ L}_{D} \; {}^t_\circ X^d{}_L \; {}^t g_{db} \; {}^\circ g^{DB} \; {}^\circ\underline{\mathbf{g}}_B \, {}^t\underline{\mathbf{g}}^b \, . \tag{2.42g}$$

Comparing Eqs. (2.42g) and (2.42c) it is obvious that:

[5] See Appendix.

${}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} = {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{T} = {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-1} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{T}$ (the last equality follows from the symmetry of ${}_{\circ}^{t}\underline{\underline{\mathbf{U}}}$).

Hence,

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-1} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-1} . \tag{2.42h} $$

Using also Eqs. (2.35) and (2.39), we obtain

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-1} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{C}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-1} = {}^{\circ}\underline{\underline{\mathbf{g}}} \tag{2.42i} $$

and the first equality (i) is shown to be correct.

To prove the second equality (ii), we write (Marsden & Hughes 1983):

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} = {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot \left({}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \right) \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} = \left({}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \right)^{2} , \tag{2.43} $$

and since ${}_{\circ}^{t}\underline{\underline{\mathbf{R}}}$ cannot be a singular matrix, the second equality (ii) is shown to be correct.

We will now show that the right polar decomposition is *unique*.

Assuming that it is not unique, we can have, together with Eq. (2.40), another decomposition, for example:

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} = {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{R}}}} \cdot {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{U}}}} , \tag{2.44a} $$

where ${}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{R}}}}$ is an orthogonal tensor and ${}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{U}}}}$ is a symmetric tensor.

We can write

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{C}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} = {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{U}}}} \cdot {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{U}}}} , \tag{2.44b} $$

and therefore,

$$ {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{U}}}} = \left[{}_{\circ}^{t}\underline{\underline{\mathbf{C}}} \right]^{1/2} . \tag{2.44c} $$

However, comparing the above with Eq. (2.39), we conclude that:

$$ {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{U}}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{U}}} . \tag{2.44d} $$

Then, from Eq. (2.44a),

$$ {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{R}}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}^{-1} , \tag{2.44e} $$

and comparing the above with Eq. (2.41), we conclude that:

$$ {}_{\circ}^{t}\underline{\underline{\tilde{\mathbf{R}}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} . \tag{2.44f} $$

Equations (2.44d) and (2.44f) show that the *right polar decomposition is unique.*

2.7.3 The Finger deformation tensor

The *Finger deformation tensor*, also known in the literature as the *left Cauchy-Green deformation tensor*, is defined at a point χ as:

$$ {}^{t}\underline{\underline{\mathbf{b}}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \,. \tag{2.45a} $$

Using Eq. (2.28c), we have

$$ {}^{t}\underline{\underline{\mathbf{b}}} \;=\; \left[{}^{t}_{\circ}X^{d}{}_{D} \; {}^{t}\underline{\mathbf{g}}_{d} \; {}^{\circ}\underline{\mathbf{g}}^{D} \right] \cdot \left[{}^{t}_{\circ}X^{a}{}_{A} \; {}^{t}g_{ab} \; {}^{\circ}g^{BA} \; {}^{\circ}\underline{\mathbf{g}}_{B} \; {}^{t}\underline{\mathbf{g}}^{b} \right] \tag{2.45b} $$

and therefore,

$$ {}^{t}\underline{\underline{\mathbf{b}}} \;=\; {}^{t}_{\circ}X^{d}{}_{B} \; {}^{t}_{\circ}X^{a}{}_{A} \; {}^{t}g_{ab} \; {}^{\circ}g^{BA} \; {}^{t}\underline{\mathbf{g}}_{d} \; {}^{t}\underline{\mathbf{g}}^{b} \,. \tag{2.45c} $$

It is important to note that the Finger deformation tensor is *defined in the spatial configuration.*

Proceeding in the same way as in Sect. 2.7.1, we can show that:

- ${}^{t}\underline{\underline{\mathbf{b}}}$ is a *symmetric* tensor.
- ${}^{t}\underline{\underline{\mathbf{b}}}$ is a *positive-definite* tensor.

2.7.4 The left polar decomposition

We define the *left polar decomposition* as a *multiplicative decomposition* of the tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}$ into a *symmetric* tensor $({}^{t}_{\circ}\underline{\underline{\mathbf{V}}})$ postmultiplied by the *orthogonal* tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{R}}}$. Therefore,

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \,. \tag{2.46} $$

From the above equation,

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \tag{2.47a} $$

and taking into account that ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ is symmetric, we get

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{T} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \,. \tag{2.47b} $$

The above equation shows that the tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$, known as the *left stretch tensor*, is *symmetric.*

From Eq. (2.45a), we get

$$ {}^{t}\underline{\underline{\mathbf{b}}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \tag{2.48a} $$

and therefore,

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \;=\; \left[\, {}^{t}\underline{\underline{\mathbf{b}}} \, \right]^{1/2} \,. \tag{2.48b} $$

From the above equation we conclude that the left stretch tensor is defined in the *spatial configuration* and that it inherits from ${}^{t}\underline{\underline{\mathbf{b}}}$ the positive definiteness.

Proceeding in the same way as in Sect. 2.7.2 we can show that the *left polar decomposition is unique.*

2.7.5 Physical interpretation of the tensors ${}^t_\circ\underline{\underline{\mathbf{R}}}$, ${}^t_\circ\underline{\underline{\mathbf{U}}}$ and ${}^t_\circ\underline{\underline{\mathbf{V}}}$

In this Section, we will discuss a physical interpretation of the second-order tensors introduced by the polar decomposition.

The rotation tensor

Assuming a motion in which ${}^t_\circ\underline{\underline{\mathbf{U}}} = {}^\circ\underline{\underline{\mathbf{g}}}$ and therefore ${}^t_\circ\underline{\underline{\mathbf{V}}} = {}^t\underline{\underline{\mathbf{g}}}$,we get

$$ {}^t_\circ\underline{\underline{\mathbf{X}}} = {}^t_\circ\underline{\underline{\mathbf{R}}} \tag{2.49}$$

and considering in the reference configuration, at the point under analysis, two arbitrary vectors ${}^\circ\mathrm{d}\underline{\mathbf{x}}_1$ and ${}^\circ\mathrm{d}\underline{\mathbf{x}}_2$ we have, in the spatial configuration:

$$ {}^t\mathrm{d}\underline{\mathbf{x}}_1 = {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_1 \tag{2.50a}$$

$$ {}^t\mathrm{d}\underline{\mathbf{x}}_2 = {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_2 \tag{2.50b}$$

hence, in the spatial configuration we can write

$$ {}^t\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^t\mathrm{d}\underline{\mathbf{x}}_2 = {}^t_\circ R^a{}_A \; {}^t_\circ R^b{}_B \; {}^t g_{ab} \; {}^\circ\mathrm{d}x_1^A \; {}^\circ\mathrm{d}x_2^B \; . \tag{2.50c}$$

Using Eqs. (2.42a) and (2.42b) we can rewrite the above equation as

$$ {}^t\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^t\mathrm{d}\underline{\mathbf{x}}_2 = {}^t_\circ R^a{}_A \; ({}^t_\circ R^T)^C{}_a \; {}^\circ\mathrm{d}x_1^A \; {}^\circ\mathrm{d}x_{2C} \; , \tag{2.50d}$$

and since ${}^t_\circ\underline{\underline{\mathbf{R}}}$ is an orthogonal tensor,

$$ {}^t\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^t\mathrm{d}\underline{\mathbf{x}}_2 = \delta^C{}_A \; {}^\circ\mathrm{d}x_1^A \; {}^\circ\mathrm{d}x_{2C} = {}^\circ\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_2 \; . \tag{2.51}$$

The above equation shows that when ${}^t_\circ\underline{\underline{\mathbf{X}}} = {}^t_\circ\underline{\underline{\mathbf{R}}}$:

- The corresponding vectors in the spatial and reference configuration have the same modulus.
- The angle between two vectors in the spatial configuration equals the angle between the corresponding two vectors in the reference configuration.

Hence, the motion can be characterized, at the point under analysis, as a rigid body rotation.

We can generalize Eqs.(2.50a-2.50d) for any vector $\underline{\mathbf{Y}}$ that in the reference configuration is associated to the point under analysis. For the particular motion described by ${}^t_\circ\underline{\underline{\mathbf{X}}} = {}^t_\circ\underline{\underline{\mathbf{R}}}$, we get

$$ {}^t\underline{\mathbf{y}} = {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot \underline{\mathbf{Y}} \; , \tag{2.52a}$$

and since the rotation tensor is orthogonal,

$$ \underline{\mathbf{Y}} = {}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot {}^t\underline{\mathbf{y}} \; . \tag{2.52b}$$

If in the reference configuration there is a relation of the form:

$$ \underline{\mathbf{Y}} = \underline{\underline{\mathbf{A}}} \cdot \underline{\mathbf{W}} \; , \tag{2.53a}$$

where,

$\underline{\mathbf{Y}}$, $\underline{\mathbf{W}}$: vectors defined in the reference configuration,
$\underline{\underline{\mathbf{A}}}$: second order tensor defined in the reference configuration,

it is easy to show that:

$$ {}^{t}\underline{\mathbf{y}} \;=\; \left[{}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\underline{\mathbf{A}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \right] \cdot {}^{t}\underline{\mathbf{w}} \; . \tag{2.53b} $$

In the above equation, the term between brackets is the result (in the spatial configuration) of the rotation of the material tensor $\underline{\underline{\mathbf{A}}}$.

If $\underline{\underline{\mathbf{A}}}$ is a symmetric second-order tensor we can write it using its eigenvalues and eigenvectors,

$$ \underline{\underline{\mathbf{A}}} \;=\; \lambda^{AB} \; \underline{\boldsymbol{\Phi}}_A \; \underline{\boldsymbol{\Phi}}_B \; , \tag{2.54a} $$

where,

$$ \lambda^{AB} = 0 \quad \text{if} \;\; A \neq B \; , $$

and the set of vectors $\underline{\boldsymbol{\Phi}}_A$ form an *orthogonal basis in the reference configuration.*

In the spatial configuration we get, from the rotation of $\underline{\underline{\mathbf{A}}}$:

$$ {}^{t}\underline{\underline{\mathbf{a}}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\underline{\mathbf{A}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \; , \tag{2.54b} $$

and therefore,

$$ {}^{t}\underline{\underline{\mathbf{a}}} \;=\; \lambda^{AB} \; \left({}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\boldsymbol{\Phi}}_A \right) \; \left(\underline{\boldsymbol{\Phi}}_B \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \right) . \tag{2.54c} $$

We now define in the spatial configuration the set of vectors ${}^{t}\underline{\boldsymbol{\varphi}}_a \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\boldsymbol{\Phi}}_A$, which obviously constitute an orthogonal basis; using Eq. (2.28a) we obtain

$$ \underline{\boldsymbol{\Phi}}_B \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t}\underline{\boldsymbol{\varphi}}_b \right) \;=\; \left({}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\boldsymbol{\Phi}}_B \right) \cdot {}^{t}\underline{\boldsymbol{\varphi}}_b \; , \tag{2.54d} $$

and therefore,

$$ {}^{t}\underline{\underline{\mathbf{a}}} \;=\; \lambda^{AB} \; \left({}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\boldsymbol{\Phi}}_A \right) \left({}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot \underline{\boldsymbol{\Phi}}_B \right) . \tag{2.54e} $$

Comparing Eqs. (2.54a) and (2.54e), we conclude that:

- The material tensor $\underline{\underline{\mathbf{A}}}$ and the spatial tensor ${}^{t}\underline{\underline{\mathbf{a}}}$ have the *same eigenvalues.*
- The *eigenvectors* of ${}^{t}\underline{\underline{\mathbf{a}}}$ are *obtained by rotating* with ${}^{t}_{\circ}\underline{\underline{\mathbf{R}}}$ the *eigenvectors* of $\underline{\underline{\mathbf{A}}}$.

Since, according to Eq. (2.47a) ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T}$, we can assess that:

- The material tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ and the spatial tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$ have the same eigenvalues.
- The eigenvectors of ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$ (and ${}^{t}\underline{\underline{\mathbf{b}}}$) are obtained by rotating with ${}^{t}_{\circ}\underline{\underline{\mathbf{R}}}$ the eigenvectors of ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ (and ${}^{t}_{\circ}\underline{\underline{\mathbf{C}}}$).

The right stretch tensor

To study the physical interpretation of the right stretch tensor we consider, in the reference configuration, at the point χ under analysis, two vectors ${}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1$ and ${}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2$ that in the spatial configuration are transformed into ${}^{t}\mathrm{d}\underline{\mathbf{x}}_1$ and ${}^{t}\mathrm{d}\underline{\mathbf{x}}_2$

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \tag{2.55a} $$

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 \ . \tag{2.55b} $$

After some algebra,

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \right) \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 \ , \tag{2.56a} $$

and using Eqs. (2.35) and (2.39), we write

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \right) \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 \ . \tag{2.56b} $$

It follows from the above equation that the changes in lengths and angles, produced by the motion, are directly associated to the right stretch tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$. In the previous subsection we showed that these changes are nil when ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}} = {}^{\circ}\underline{\underline{\mathbf{g}}}$.

The left stretch tensor

Starting from Eqs. (2.55a-2.55b) and using a left polar decomposition, we get

$$ {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{-1} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{-1} \right) \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 \ . \tag{2.57} $$

It is obvious from the above equation that changes in lengths and angles, produced by the motion, are directly associated to the left stretch tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$. Above we showed that those changes are nil when ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}} = {}^{t}\underline{\underline{\mathbf{g}}}$.

2.7.6 Numerical algorithm for the polar decomposition

When analyzing finite element models of nonlinear solid mechanics problems, we usually know the numerical value of the deformation gradient tensor at a point and we need to use a numerical algorithm for performing the polar decomposition.

In what follows, we present an algorithm that can be used for the right polar decomposition when we refer the problem to a fixed Cartesian system.

√ Starting from the matrix $\left[{}^{t}_{\circ}X\right]$ that is a (3×3)-matrix in the general case, we calculate the symmetric matrix,

$$ \left[{}^{t}_{\circ}C\right] = \left[{}^{t}_{\circ}X\right]^{T} \left[{}^{t}_{\circ}X\right] \ . \tag{2.58a} $$

- √ Using a numerical algorithm (Bathe 1996), we calculate the eigenvalues λ_A^2 and eigenvectors $[\Phi_A]$; $A = 1, 2, 3$ of the matrix $[{}^t_\circ C]$.
- √ From the above step,

$$[{}^t_\circ C] = [\Psi]\ [\Lambda]\ [\Psi]^T \tag{2.58b}$$

where

$$[\Psi] = [[\Phi_1] \quad [\Phi_2] \quad [\Phi_3]] \tag{2.58c}$$

and

$$[\Lambda] = \begin{bmatrix} (\lambda_1)^2 & 0 & 0 \\ 0 & (\lambda_2)^2 & 0 \\ 0 & 0 & (\lambda_3)^2 \end{bmatrix} . \tag{2.58d}$$

- √ Using Eq. (2.39), we obtain (Strang 1980)

$$[{}^t_\circ U] = [\Psi]\ [\Lambda]^{1/2}\ [\Psi]^T \tag{2.58e}$$

where

$$[\Lambda]^{\frac{1}{2}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} . \tag{2.58f}$$

- √ Finally, using Eq. (2.40) we get

$$[{}^t_\circ R] = [{}^t_\circ X]\ [{}^t_\circ U]^{-1} . \tag{2.58g}$$

And $[{}^t_\circ R]$ is now also completely determined.

Example 2.7. ◀◀◀◀◀

For the case analyzed in Example 2.4, and for $\gamma = 15°$ we can write

$$[{}^t_\circ X] = \begin{bmatrix} 1.0 & 0.26795 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} .$$

Hence,

$$\begin{bmatrix} {}^t_\circ C \end{bmatrix} = \begin{bmatrix} {}^t_\circ X^T \end{bmatrix} \begin{bmatrix} {}^t_\circ X \end{bmatrix} = \begin{bmatrix} 1.0 & 0.26795 & 0.0 \\ 0.26795 & 1.0718 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} .$$

Solving for the eigenvalues and eigenvectors of $\begin{bmatrix} {}^t_\circ C \end{bmatrix}$, we get

$$[\Lambda] = \begin{bmatrix} 0.76556 & 0.0 & 0.0 \\ 0.0 & 1.30624 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} ,$$

and,

$$[\Psi] = \begin{bmatrix} -0.75259 & 0.65849 & 0.0 \\ 0.65849 & 0.75259 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} .$$

Using Eq. (2.58e), we write

$$\begin{bmatrix} {}^t_\circ U \end{bmatrix} = [\Psi] \; [\Lambda]^{1/2} \; [\Psi]^T = \begin{bmatrix} 0.99114 & 0.13279 & 0.0 \\ 0.13279 & 1.02672 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} ,$$

and finally, using Eq. (2.58g),

$$\begin{bmatrix} {}^t_\circ R \end{bmatrix} = \begin{bmatrix} {}^t_\circ X \end{bmatrix} \begin{bmatrix} {}^t_\circ U \end{bmatrix}^{-1} = \begin{bmatrix} 0.99114 & 0.13279 & 0.0 \\ -0.13279 & 0.99114 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} .$$

We urge the reader to verify that:
- $\begin{bmatrix} {}^t_\circ R \end{bmatrix}^T \begin{bmatrix} {}^t_\circ R \end{bmatrix} = [I]$
- $\begin{bmatrix} {}^t_\circ X \end{bmatrix} = \begin{bmatrix} {}^t_\circ R \end{bmatrix} \begin{bmatrix} {}^t_\circ U \end{bmatrix}$

within the *numerical accuracy* used in the above calculations. ◀◀◀◀◀

Example 2.8. ◀◀◀◀◀

For the case analyzed in the previous example, we are now going to describe the deformation of a material fiber that in the reference configuration contains the $(0,0,0)$ point and has the direction $[{}^\circ n]^T = \begin{bmatrix} 0.0 & 1.0 & 0.0 \end{bmatrix}$. In the reference configuration,

$${}^\circ \mathrm{d}\underline{\mathbf{x}} = {}^\circ \mathrm{d}S \; {}^\circ \underline{\mathbf{n}}$$

$${}^\circ \mathrm{d}\underline{\mathbf{x}} \cdot {}^\circ \mathrm{d}\underline{\mathbf{x}} = ({}^\circ \mathrm{d}S)^2 \quad .$$

In the spatial configuration,

$${}^t \mathrm{d}\underline{\mathbf{x}} = {}^t \mathrm{d}S \; {}^t \underline{\mathbf{n}} \quad ; \quad \|{}^t \underline{\mathbf{n}}\| = 1$$

$${}^t \mathrm{d}\underline{\mathbf{x}} \cdot {}^t \mathrm{d}\underline{\mathbf{x}} = ({}^t \mathrm{d}S)^2 \, .$$

Using Eq. (2.56b),

$$({}^t\mathrm{d}S)^2 \;=\; (\,{}^\circ\mathrm{d}S)^2 \quad \left[\,{}^\circ\underline{\mathbf{n}} \cdot {}^t_\circ\underline{\underline{\mathbf{C}}} \cdot {}^\circ\underline{\mathbf{n}}\right]$$

and therefore, in the fixed Cartesian system that we are using,

$$\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S} \;=\; \left[\,[{}^\circ n]^T\;[{}^t_\circ C]\;[{}^\circ n]\,\right]^{1/2}$$

and using the numerical values calculated in the previous example,

$$\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S} \;=\; 1.03528\ .$$

The reader can check the above numerical values using very simple geometrical considerations.

∘ In the case of ${}^\circ\underline{\mathbf{n}} \;=\; \underline{\mathbf{\Phi}}_i$, it is easy to show that

$$\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S} \;=\; \lambda_i\ .$$

∘ For $[{}^\circ m]^T = \;\begin{bmatrix}1.0\ 0.0\ 0.0\end{bmatrix}$, we get

$$\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S} \;=\; 1.0\ .$$

∘ The vectors ${}^\circ\underline{\mathbf{m}}$ and ${}^\circ\underline{\mathbf{n}}$, which are orthogonal in the reference configuration, form an angle ${}^t\theta$ in the spatial configuration,

$$\begin{aligned}
{}^t\mathrm{d}\underline{\mathbf{x}}_m &= {}^t\mathrm{d}S_m\,{}^t\underline{\mathbf{m}} \qquad ; \qquad \|{}^t\underline{\mathbf{m}}\| = 1\\
{}^t\mathrm{d}\underline{\mathbf{x}}_n &= {}^t\mathrm{d}S_n\;{}^t\underline{\mathbf{n}} \qquad ; \qquad \|{}^t\underline{\mathbf{n}}\| = 1
\end{aligned}$$

$${}^t\mathrm{d}\underline{\mathbf{x}}_m \cdot {}^t\mathrm{d}\underline{\mathbf{x}}_n \;=\; ({}^t\mathrm{d}S_m)\;({}^t\mathrm{d}S_n)\;\cos{}^t\theta \;= (\,{}^\circ\mathrm{d}S_m)\;\;(\,{}^\circ\mathrm{d}S_n)\;\;[{}^\circ m]^T\;\;[{}^t_\circ C]\;\;[{}^\circ n]$$

$$\left(\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S}\right)_m \;\left(\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S}\right)_n \quad \cos{}^t\theta = [{}^\circ m]^T\;\;[{}^t_\circ C]\;\;[{}^\circ n]$$

hence,

$${}^t\theta \;=\; \cos^{-1}\left[\frac{[{}^\circ m]^T\;\;[{}^t_\circ C]\;\;[{}^\circ n]}{\left(\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S}\right)_m\;\;\left(\frac{{}^t\mathrm{d}S}{{}^\circ\mathrm{d}S}\right)_n}\right]$$

and using the calculated numerical values, we get

$${}^t\theta \;=\; 75^\circ\ .$$

Once again, it is very simple for the reader to check the above numerical result. ◀◀◀◀◀

In the above examples we have numerically calculated the eigenvalues and eigenvectors of the tensor ${}^t_\circ\underline{\underline{\mathbf{C}}}$; however, in some problems it is necessary to

differentiate those eigenvalues and eigenvectors and it is therefore necessary to use an analytical expression of them.

As is wellknown the eigenvalues of ${}^t_{\circ}\underline{\underline{\mathbf{C}}}$ are given by the roots of the following polynomial (Strang 1980),

$$p(\lambda_i^2) \;=\; -\;\lambda_i^6 \;+\; I_1^C\;\lambda_i^4 \;-\; I_2^C\;\lambda_i^2 \;+\; I_3^C \;=\; 0 \qquad (i=1,2,3) \tag{2.59a}$$

where, (McConnell 1957)

$$I_1^C \;=\; tr({}^t_{\circ}\underline{\underline{\mathbf{C}}}) \tag{2.59b}$$

$$I_2^C \;=\; \frac{1}{2}\;\left[(I_1^C)^2 \;-\; tr({}^t_{\circ}\underline{\underline{\mathbf{C}}}^2)\right] \tag{2.59c}$$

$$I_3^C \;=\; det({}^t_{\circ}\underline{\underline{\mathbf{C}}})\;, \tag{2.59d}$$

and it is easy to verify the *Serrin representation* (Simo & Taylor 1991):

$${}^t\underline{\boldsymbol{\varphi}}_a\;{}^t\underline{\boldsymbol{\varphi}}_a \;=\; {}^t\lambda_a^2\;\frac{{}^t\underline{\underline{\mathbf{b}}} -\;(I_1^C-\lambda_a^2)\;{}^t\underline{\underline{\mathbf{g}}}\;+\;I_3^C\;\lambda_a^{-2}\;{}^t\underline{\underline{\mathbf{b}}}^{-1}}{2\;\lambda_a^4\;-\;I_1^C\;\lambda_a^2\;+\;I_3^C\;\lambda_a^{-2}}\;, \tag{2.60a}$$

$$\underline{\boldsymbol{\Phi}}_A\;\underline{\boldsymbol{\Phi}}_A \;=\; {}^t\lambda_A^2\;\frac{{}^t_{\circ}\underline{\underline{\mathbf{C}}}\;-\;(I_1^C-\lambda_A^2)\;{}^{\circ}\underline{\underline{\mathbf{g}}}\;+\;I_3^C\;\lambda_A^{-2}\;{}^t_{\circ}\underline{\underline{\mathbf{C}}}^{-1}}{2\;\lambda_A^4\;-\;I_1^C\;\lambda_A^2\;+\;I_3^C\;\lambda_A^{-2}}\;, \tag{2.60b}$$

with no addition in "a"or "A"in the above equations. According to what we showed in Sect. 2.7.5, $\lambda_a \;=\; \lambda_A$ for $a \;=\; A$.

Example 2.9. ◀◀◀◀◀

∘ To verify Eq. (2.60a)we start from,

$$\begin{aligned}
{}^t\underline{\underline{\mathbf{b}}} &= \lambda_i^2\;{}^t\underline{\boldsymbol{\varphi}}_i\;{}^t\underline{\boldsymbol{\varphi}}_i \\
{}^t\underline{\underline{\mathbf{b}}}^{-1} &= \lambda_i^{-2}\;{}^t\underline{\boldsymbol{\varphi}}_i\;{}^t\underline{\boldsymbol{\varphi}}_i \\
I_1^C &= \lambda_1^2\;+\;\lambda_2^2\;+\;\lambda_3^2 \\
I_2^C &= \lambda_1^2\;\lambda_2^2\;+\;\lambda_1^2\;\lambda_3^2\;+\;\lambda_2^2\;\lambda_3^2 \\
I_3^C &= \lambda_1^2\;\lambda_2^2\;\lambda_3^2\;.
\end{aligned}$$

For $a=1$

$${}^t\underline{\underline{\mathbf{b}}} - (I_1^C-\lambda_1^2){}^t\underline{\underline{\mathbf{g}}}\;+\;I_3^C\;\lambda_1^{-2}{}^t\underline{\underline{\mathbf{b}}}^{-1} \;=\; \frac{2\lambda_1^4 - I_1^C\;\lambda_1^2 + I_3^C\;\lambda_1^{-2}}{\lambda_1^2}\;{}^t\underline{\boldsymbol{\varphi}}_1\;{}^t\underline{\boldsymbol{\varphi}}_1\;.$$

The above verifies Eq. (2.60a) for the case $a \;=\; 1$; the demonstrations for $a=2,3$ are identical.

∘ To verify Eq. (2.60b) we start from,

$$
\begin{aligned}
{}^t_\circ\underline{\underline{\mathbf{C}}} &= \lambda_I{}^2\ \underline{\mathbf{\Phi}}_I\ \underline{\mathbf{\Phi}}_I\\
{}^t_\circ\underline{\underline{\mathbf{C}}}^{-1} &= \lambda_I{}^{-2}\ \underline{\mathbf{\Phi}}_I\ \underline{\mathbf{\Phi}}_I
\end{aligned}
$$

and proceed as before. ◀◀◀◀◀

It is important to note that Eqs. (2.60a-2.60b) are only valid if the denominator of the r.h.s. is not zero. The denominator is zero if we have repeated eigenvalues.

2.8 Strain measures

In the literature we can find a large number of *strain measures* that have been proposed to characterize a deformation process. There are different approaches for analyzing the deformation of continuum bodies and we usually find that, for a given approach, one particular strain measure may be more suitable than others.

In this section we will present a number of these strain measures without making any claim of completeness.

2.8.1 The Green deformation tensor

We have already presented the Green deformation tensor in Sect. 2.7.1. It is important to remember that this second-order tensor is defined in the *reference configuration* and that for two vectors ${}^\circ\mathrm{d}\underline{\mathbf{x}}_1$ and ${}^\circ\mathrm{d}\underline{\mathbf{x}}_2$ defined at a point χ in the reference configuration, the corresponding vectors in the spatial configuration satisfy the relation,

$$
{}^t\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^t\mathrm{d}\underline{\mathbf{x}}_2 = {}^\circ\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^t_\circ\underline{\underline{\mathbf{C}}} \cdot {}^\circ\mathrm{d}\underline{\mathbf{x}}_2 \,. \tag{2.61}
$$

2.8.2 The Finger deformation tensor

We have already presented the Finger deformation tensor in Sect. 2.7.3. It is important to remember that this second-order tensor is defined in the *spatial configuration.*

Using Eq. (2.45a),

$$
{}^t\underline{\underline{\mathbf{b}}}^{-1} = {}^t_\circ\underline{\underline{\mathbf{X}}}^{-T} \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \,, \tag{2.62}
$$

and for two vectors ${}^t\mathrm{d}\underline{\mathbf{x}}_1$ and ${}^t\mathrm{d}\underline{\mathbf{x}}_2$ defined at a point ${}^t\underline{\mathbf{x}}$ in the spatial configuration we can write,

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\underline{\underline{\mathbf{b}}}^{-1} \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-T} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 \tag{2.63a} $$

using Eq. (2.28a), we get

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\underline{\underline{\mathbf{b}}}^{-1} \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 \ . \tag{2.63b} $$

2.8.3 The Green-Lagrange deformation tensor

From Eq.(2.61),

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 - {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = 2\ {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left[\frac{1}{2}({}^{t}_{\circ}\underline{\underline{\mathbf{C}}} - {}^{\circ}\underline{\underline{\mathbf{g}}})\right] \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 \ . \tag{2.64} $$

The *Green-Lagrange strain tensor is defined in the reference configuration* as

$$ {}^{t}_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}({}^{t}_{\circ}\underline{\underline{\mathbf{C}}} - {}^{\circ}\underline{\underline{\mathbf{g}}}) \ . \tag{2.65} $$

The second order tensor ${}^{t}_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}}$ describes the deformation corresponding to the t-configuration (spatial configuration) referred to the configuration at $t = 0$ (reference configuration).

Example 2.10. ◀◀◀◀◀
Considering a convected coordinate system $\{\theta^i\}$ with covariant base vectors ${}^{t}\underline{\widetilde{\mathbf{g}}}_i$ in the spatial configuration and ${}^{\circ}\underline{\widetilde{\mathbf{g}}}_i$ in the reference one, we can write,

$$ {}^{t}_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} = {}^{t}_{\circ}\widetilde{\varepsilon}_{lm}\ {}^{\circ}\underline{\widetilde{\mathbf{g}}}^{l}\ {}^{\circ}\underline{\widetilde{\mathbf{g}}}^{m} $$

and it is easy to show that,

$$ {}^{t}_{\circ}\widetilde{\varepsilon}_{lm} = \frac{1}{2}\left[{}^{t}\underline{\widetilde{\mathbf{g}}}_l \cdot {}^{t}\underline{\widetilde{\mathbf{g}}}_m - {}^{\circ}\underline{\widetilde{\mathbf{g}}}_l \cdot {}^{\circ}\underline{\widetilde{\mathbf{g}}}_m\right] \ . $$

◀◀◀◀◀

At the point χ under study, we now evolve from the t-configuration to a $t + \Delta t$-configuration by means of a rotation ${}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}$. Hence,

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \ , \tag{2.66a} $$

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{C}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \ , \tag{2.66b} $$

and taking into account that the rotation tensor is orthogonal, we get

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{C}}} \equiv {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} \tag{2.66c} $$

and therefore,

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} \equiv {}^{t}_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} \,. \tag{2.66d} $$

From the above, we conclude that the Green deformation tensor and the Green -Lagrange strain tensor are not affected by rigid body rotations: that is to say they are *indifferent to rotations.*

2.8.4 The Almansi deformation tensor

From Eq. (2.63b), we get

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 - {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = 2\, {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left[\frac{1}{2} ({}^{t}\underline{\underline{\mathbf{g}}} - {}^{t}\underline{\underline{\mathbf{b}}}^{-1}) \right] \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 \,. \tag{2.67} $$

The *Almansi strain tensor is defined in the spatial configuration* as

$$ {}^{t}\underline{\underline{\mathbf{e}}} = \frac{1}{2} ({}^{t}\underline{\underline{\mathbf{g}}} - {}^{t}\underline{\underline{\mathbf{b}}}^{-1}) \,. \tag{2.68} $$

At the point χ under study, we now evolve from the t-configuration to the $t + \Delta t$-configuration by means of a rotation ${}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}$. Hence,

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \,, \tag{2.69a} $$

and,

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{X}}}^{-T} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-T} \,, \tag{2.69b} $$

using Eq. (2.62),

$$ {}^{t+\Delta t}\underline{\underline{\mathbf{b}}}^{-1} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\underline{\mathbf{b}}}^{-1} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \,, \tag{2.69c} $$

and therefore,

$$ {}^{t+\Delta t}\underline{\underline{\mathbf{e}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\underline{\mathbf{e}}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \,. \tag{2.69d} $$

Hence, the *Finger and Almansi tensors are affected by rigid-body rotations.*

2.8.5 The Hencky deformation tensor

The *Hencky or logarithmic strain tensor is defined in the reference configuration* as

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{H}}} = \ln {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \,. \tag{2.70} $$

When the problem is referred to a fixed Cartesian system using Eq. (2.58e), we get

$$ [{}^{t}_{\circ}H] = [\Psi] \begin{bmatrix} \ln\lambda_1 & 0.0 & 0.0 \\ 0.0 & \ln\lambda_2 & 0.0 \\ 0.0 & 0.0 & \ln\lambda_3 \end{bmatrix} [\Psi]^{T} \,. \tag{2.71} $$

Obviously, *the Hencky deformation tensor is indifferent to rotations,* since from the polar decomposition, we can see that ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ does not incorporate the effect of rigid-body rotations.

2.9 Representation of spatial tensors in the reference configuration ("pull-back")

For the regular motion depicted in Fig. 2.1, we can define:

- An arbitrary curvilinear coordinate system $\{{}^t x^a\}$ in the spatial configuration. At a point χ $({}^t x^a, a = 1,2,3)$ we can determine the covariant base vectors ${}^t\underline{\mathbf{g}}_a$ and the contravariant base vectors ${}^t\underline{\mathbf{g}}^a$.
- An arbitrary curvilinear coordinate system $\{{}^\circ x^A\}$ in the reference configuration. At the point χ $({}^\circ x^A, A = 1,2,3)$ we can determine the covariant base vectors ${}^\circ\underline{\mathbf{g}}_A$ and the contravariant base vectors ${}^\circ\underline{\mathbf{g}}^A$.
- A convected curvilinear coordinate system $\{\theta^i\}$. At the point χ in the spatial configuration we can determine the covariant base vectors ${}^t\underline{\tilde{\mathbf{g}}}_a$ and the contravariant base vectors ${}^t\underline{\tilde{\mathbf{g}}}^a$, while in the reference configuration we can determine the covariant base vectors ${}^\circ\underline{\tilde{\mathbf{g}}}_a$ and the contravariant base vectors ${}^\circ\underline{\tilde{\mathbf{g}}}^a$

2.9.1 Pull-back of vector components

Let us consider in the spatial configuration at the point χ a vector,

$$ {}^t\underline{\mathbf{b}} = {}^t b^i \; {}^t\underline{\mathbf{g}}_i = {}^t b_i \; {}^t\underline{\mathbf{g}}^i = {}^t\tilde{b}^i \; {}^t\underline{\widetilde{\mathbf{g}}}_i = {}^t\tilde{b}_i \; {}^t\underline{\widetilde{\mathbf{g}}}^i \; . \tag{2.72} $$

We *define* in the reference configuration the following vectors (Dvorkin, Goldschmit, Pantuso & Repetto 1994):

$$ {}^t\underline{\mathbf{B}}^\sharp = {}^t\tilde{b}^i \; {}^\circ\underline{\widetilde{\mathbf{g}}}_i = [\, {}^t\underline{\mathbf{B}}^\sharp]^A \; {}^\circ\underline{\mathbf{g}}_A \tag{2.73a} $$

$$ {}^t\underline{\mathbf{B}}^\flat = {}^t\tilde{b}_i \; {}^\circ\underline{\widetilde{\mathbf{g}}}^i = [\, {}^t\underline{\mathbf{B}}^\flat]_A \; {}^\circ\underline{\mathbf{g}}^A \; . \tag{2.73b} $$

After some algebra,

$$ [{}^t\underline{\mathbf{B}}^\sharp]^A = {}^t b^j \; ({}^t_\circ X^{-1})^A_{\ j} \tag{2.74a} $$

$$ [{}^t\underline{\mathbf{B}}^\flat]_A = {}^t b_j \; {}^t_\circ X^j_{\ A} \; . \tag{2.74b} $$

Adopting the notation used in manifolds analysis (Lang 1972, Marsden & Hughes 1983) we define the *pull-back of the contravariant components* ${}^t b^j$ as

$$ \left[{}^t\phi^*({}^t b^j)\right]^A = [{}^t\underline{\mathbf{B}}^\sharp]^A \tag{2.75a} $$

and the *pull-back of the covariant components* ${}^t b_j$ as:

$$ \left[{}^t\phi^*({}^t b_j)\right]_A = [{}^t\underline{\mathbf{B}}^\flat]_A \; . \tag{2.75b} $$

We can therefore rewrite Eqs. (2.73a-2.73b) and (2.74a-2.74b),

$$
\begin{aligned}
{}^t\underline{\mathbf{B}}^\sharp &= {}^tb^j\ ({}^t_\circ X^{-1})^A_{\ j}\ {}^\circ\underline{\mathbf{g}}_A = {}^tb^j\ ({}^t_\circ X^{-1})^A_{\ j}\ {}^\circ g_{AC}\ {}^\circ\underline{\mathbf{g}}^C && (2.76a)\\
{}^t\underline{\mathbf{B}}^\flat &= {}^tb_j\ {}^t_\circ X^j_{\ A}\ {}^\circ\underline{\mathbf{g}}^A = {}^tb_j\ {}^t_\circ X^j_{\ A}\ {}^\circ g^{AC}\ {}^\circ\underline{\mathbf{g}}_C\,. && (2.76b)
\end{aligned}
$$

For two vectors ${}^t\underline{\mathbf{b}}$ and ${}^t\underline{\mathbf{w}}$, defined in the spatial configuration at χ, using Eqs. (2.74a-2.74b), it is easy to show that:

$$
{}^t\underline{\mathbf{B}}^\sharp \cdot {}^t\underline{\mathbf{W}}^\flat = {}^t\underline{\mathbf{B}}^\flat \cdot {}^t\underline{\mathbf{W}}^\sharp = {}^t\underline{\mathbf{b}} \cdot {}^t\underline{\mathbf{w}}\,. \tag{2.77}
$$

Also, using Eqs.(2.74a-2.74b) and (2.36a-2.36b), we get

$$
[{}^t\underline{\mathbf{B}}^\sharp]^B\ {}^t_\circ C_{AB} = [{}^t\underline{\mathbf{B}}^\flat]_A\,. \tag{2.78}
$$

Using Eqs. (2.73a-2.73b), we can write

$$
\begin{aligned}
({}^t\underline{\widetilde{\mathbf{g}}}_a)^\sharp &= {}^\circ\underline{\widetilde{\mathbf{g}}}_a && (2.79a)\\
({}^t\underline{\widetilde{\mathbf{g}}}^a)^\flat &= {}^\circ\underline{\widetilde{\mathbf{g}}}^a\,. && (2.79b)
\end{aligned}
$$

Hence, we use the following notation (Moran, Ortiz & Shih 1990):

$$
\begin{aligned}
{}^t\phi^*({}^t\underline{\widetilde{\mathbf{g}}}_a) &= ({}^t\underline{\widetilde{\mathbf{g}}}_a)^\sharp = {}^\circ\underline{\widetilde{\mathbf{g}}}_a && (2.79c)\\
{}^t\phi^*({}^t\underline{\widetilde{\mathbf{g}}}^a) &= ({}^t\underline{\widetilde{\mathbf{g}}}^a)^\flat = {}^\circ\underline{\widetilde{\mathbf{g}}}^a\,. && (2.79d)
\end{aligned}
$$

From the above equations, we can get by inspection the geometrical interpretation of the vectors ${}^t\underline{\mathbf{B}}^\sharp$ and ${}^t\underline{\mathbf{B}}^\flat$:

- If ${}^t\underline{\mathbf{b}}$, in the spatial configuration, is the tangent to a curve ${}^t\underline{\mathbf{c}}(\xi)$ at a point χ, then ${}^t\underline{\mathbf{B}}^\sharp$ is the tangent, in the reference configuration, to the curve $\underline{\mathbf{C}}(\xi) = {}^t\underline{\boldsymbol{\phi}}^{-1}[{}^t\underline{\mathbf{c}}(\xi)]$, at the point χ.
- In the transformation ${}^t\underline{\mathbf{B}}^\sharp \to {}^t\underline{\mathbf{b}}$ the modulus of the original vector gets stretched as the material fiber to which they are tangent.
  In convected coordinates we have

$$
\begin{aligned}
{}^t\mathrm{d}\underline{\mathbf{x}} &= \mathrm{d}\theta^i\ {}^t\underline{\widetilde{\mathbf{g}}}_i && (2.80a)\\
{}^\circ\mathrm{d}\underline{\mathbf{x}} &= \mathrm{d}\theta^i\ {}^\circ\underline{\widetilde{\mathbf{g}}}_i && (2.80b)
\end{aligned}
$$

  that is to say,

$$
{}^t\mathrm{d}\underline{\mathbf{X}}^\sharp = {}^\circ\mathrm{d}\underline{\mathbf{x}}\,. \tag{2.80c}
$$

- For two vectors ${}^t\underline{\mathbf{b}}$ and ${}^t\underline{\mathbf{w}}$ that are orthogonal in the spatial configuration it is obvious that:

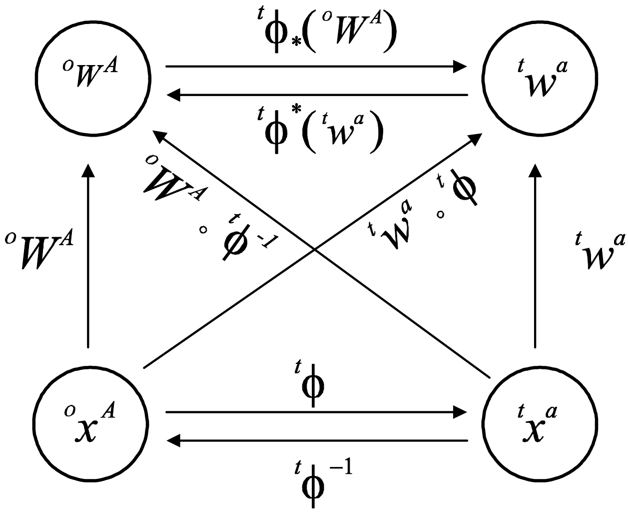


Fig. 2.4. Mappings

$$ {}^t\underline{\mathbf{B}}^\sharp \cdot {}^t\underline{\mathbf{W}}^\flat = {}^t\underline{\mathbf{B}}^\flat \cdot {}^t\underline{\mathbf{W}}^\sharp = {}^t\underline{\mathbf{b}} \cdot {}^t\underline{\mathbf{w}} = 0 \,. \tag{2.81} $$

Hence, the orthogonality of ${}^t\underline{\mathbf{b}}$ and ${}^t\underline{\mathbf{w}}$ implies the orthogonality of ${}^t\underline{\mathbf{B}}^\sharp$ and ${}^t\underline{\mathbf{W}}^\flat$ and the orthogonality of ${}^t\underline{\mathbf{B}}^\flat$ and ${}^t\underline{\mathbf{W}}^\sharp$ in the reference configuration.

It is important to take into account that in Eqs. (2.74a-2.74b) the terms on the r.h.s. must be written as a function of the coordinates in the reference configuration. We can indicate this using a more formal nomenclature (e.g. Marsden & Hughes 1983),

$$ [{}^t\underline{\mathbf{B}}^\sharp]^A = \left[{}^t\phi^*({}^tb^a)\right]^A = \left[({}^t_\circ X^{-1})^A_{\ a} \circ {}^t\phi\right] \left({}^tb^a \circ {}^t\phi\right) \tag{2.82a} $$

$$ [{}^t\underline{\mathbf{B}}^\flat]_A = \left[{}^t\phi^*({}^tb_a)\right]_A = {}^t_\circ X^a_{\ A} \left({}^tb_a \circ {}^t\phi\right) . \tag{2.82b} $$

In order to understand the above equations we use Fig. 2.4 (Marsden & Hughes 1983). In this figure we indicate with "$f \circ g$" the composition of the mapping "g" followed by the mapping "f".

Example 2.11. ◀◀◀◀◀

For a function $f({}^tx^a)$ defined in the spatial configuration, we can write:

$$\mathrm{d}f \;=\; \frac{\partial f}{\partial^t x^a}\;{}^t\mathrm{d}x^a \;.$$

In the above equation, we use a formal analogy with vector calculus in which (Marsden & Hughes 1983): $\mathrm{d}f$ is a vector; $\frac{\partial f}{\partial^t x^a}$ are its covariant components $(\mathrm{d}f_a)$ and $d^t x^a$ are contravariant base vectors.
Hence we can do a pull-back operation,

$$\left[{}^t\phi^*(\mathrm{d}f_a)\right]_A \;=\; {}^t_\circ X^a{}_A\;\mathrm{d}f_a \;=\; \frac{\partial^t x^a}{\partial^\circ x^A}\,\frac{\partial f}{\partial^t x^a} \;=\; \frac{\partial f}{\partial^\circ x^A}\;.$$

Using a more formal nomenclature and the mappings in Fig. 2.7, we get

$$\left[{}^t\phi^*(\mathrm{d}f_a)\right]_A \;=\; \frac{\partial(f\;\circ\;{}^t\phi)}{\partial^\circ x_A}\;.$$

Example 2.12.
Equation (2.11b) defines, in the spatial configuration, the velocity of a material point,

$${}^t\underline{\mathbf{v}} \;=\; {}^t v^a\;{}^t\underline{\mathbf{g}}_a\;.$$

Using the expressions for the pull-back of contravariant components, we write

$$\left[{}^t\phi^*({}^t v^a)\right]^A \;=\; ({}^t_\circ X^{-1})^A_{\;a}\;{}^t v^a\;.$$

If the coordinate system $\{{}^\circ x^A\}$, defined in the reference configuration, is a convected system with covariant base vectors, ${}^t\underline{\tilde{\mathbf{g}}}_A$, in the spatial configuration, we can write

$${}^\circ\mathrm{d}x^A\;{}^t\underline{\tilde{\mathbf{g}}}_A \;=\; {}^t\mathrm{d}x^a\;{}^t\underline{\mathbf{g}}_a$$

hence,

$${}^t\underline{\mathbf{g}}_a \;=\; ({}^t_\circ X^{-1})^A_{\;a}\;{}^t\underline{\tilde{\mathbf{g}}}_A$$

and therefore,

$${}^t\underline{\mathbf{v}} \;=\; {}^t v^a\;({}^t_\circ X^{-1})^A_{\;a}\;{}^t\underline{\tilde{\mathbf{g}}}_A\;.$$

The components of the material velocity vector in the convected system $\{{}^\circ x^A\}$ are,

$${}^t\tilde{v}^A \;=\; {}^t v^a\;({}^t_\circ X^{-1})^A_{\;a}$$

and therefore

$$\left[{}^t\phi^*({}^t v^a)\right]^A \;=\; {}^t\tilde{v}^A\;.$$

2.9.2 Pull-back of tensor components

Let us consider in the spatial configuration at the point χ $({}^t x^a, a = 1, 2, 3)$ a second-order tensor,

$$\begin{aligned} {}^t\underline{\underline{\mathbf{t}}} = {}^t t^{ij}\ {}^t\underline{\mathbf{g}}_i\ {}^t\underline{\mathbf{g}}_j = {}^t t_{ij}\ {}^t\underline{\mathbf{g}}^i\ {}^t\underline{\mathbf{g}}^j = {}^t t^i{}_j\ {}^t\underline{\mathbf{g}}_i\ {}^t\underline{\mathbf{g}}^j \\ = {}^t \tilde{t}^{ij}\ {}^t\underline{\tilde{\mathbf{g}}}_i\ {}^t\underline{\tilde{\mathbf{g}}}_j = {}^t\tilde{t}_{ij}\ {}^t\underline{\tilde{\mathbf{g}}}^i\ {}^t\underline{\tilde{\mathbf{g}}}^j = {}^t\tilde{t}^i{}_j\ {}^t\underline{\tilde{\mathbf{g}}}_i\ {}^t\underline{\tilde{\mathbf{g}}}^j \end{aligned} . \tag{2.83}$$

We *define* in the reference configuration the following second-order tensors (Dvorkin, Goldschmit, Pantuso & Repetto 1994):

$${}^t\underline{\underline{\mathbf{T}}}^\sharp = {}^t\tilde{t}^{ij}\ {}^\circ\underline{\tilde{\mathbf{g}}}_i\ {}^\circ\underline{\tilde{\mathbf{g}}}_j = [\ {}^t\underline{\underline{\mathbf{T}}}^\sharp]^{AB}\ {}^\circ\underline{\mathbf{g}}_A\ {}^\circ\underline{\mathbf{g}}_B , \tag{2.84a}$$

$${}^t\underline{\underline{\mathbf{T}}}^\natural = {}^t\tilde{t}^i{}_j\ {}^\circ\underline{\tilde{\mathbf{g}}}_i\ {}^\circ\underline{\tilde{\mathbf{g}}}^j = [\ {}^t\underline{\underline{\mathbf{T}}}^\natural]^A{}_B\ {}^\circ\underline{\mathbf{g}}_A\ {}^\circ\underline{\mathbf{g}}^B , \tag{2.84b}$$

$${}^t\underline{\underline{\mathbf{T}}}^\flat = {}^t\tilde{t}_{ij}\ {}^\circ\underline{\tilde{\mathbf{g}}}^i\ {}^\circ\underline{\tilde{\mathbf{g}}}^j = [\ {}^t\underline{\underline{\mathbf{T}}}^\flat]_{AB}\ {}^\circ\underline{\mathbf{g}}^A\ {}^\circ\underline{\mathbf{g}}^B . \tag{2.84c}$$

After some algebra we get

$$[{}^t\underline{\underline{\mathbf{T}}}^\sharp]^{AB} = {}^t t^{ab}\ ({}^t_\circ X^{-1})^A_{\ a}\ ({}^t_\circ X^{-1})^B_{\ b} , \tag{2.85a}$$

$$[{}^t\underline{\underline{\mathbf{T}}}^\natural]^A{}_B = {}^t t^a{}_b\ ({}^t_\circ X^{-1})^A_{\ a}\ ({}^t_\circ X)^b{}_B , \tag{2.85b}$$

$$[{}^t\underline{\underline{\mathbf{T}}}^\flat]_{AB} = {}^t t_{ab}\ ({}^t_\circ X)^a{}_A\ ({}^t_\circ X)^b{}_B . \tag{2.85c}$$

We define:

- *Pull-back of the contravariant components* of ${}^t\underline{\underline{\mathbf{t}}}$,

$$\left[{}^t\phi^*({}^t t^{ij})\right]^{AB} = [{}^t\underline{\underline{\mathbf{T}}}^\sharp]^{AB} . \tag{2.86a}$$

- *Pull-back of the mixed components* of ${}^t\underline{\underline{\mathbf{t}}}$,

$$\left[{}^t\phi^*({}^t t^i{}_j)\right]^A{}_B = [{}^t\underline{\underline{\mathbf{T}}}^\natural]^A{}_B . \tag{2.86b}$$

- *Pull-back of the covariant components* of ${}^t\underline{\underline{\mathbf{t}}}$,

$$\left[{}^t\phi^*({}^t t_{ij})\right]_{AB} = [{}^t\underline{\underline{\mathbf{T}}}^\flat]_{AB} . \tag{2.86c}$$

From Eqs. (2.84a-2.84c) and (2.85a-2.85c) we can write,

$$\begin{aligned}
{}^{t}\underline{\underline{\mathbf{T}}}^{\sharp} &= {}^{t}t^{ab}\ ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X^{-1})^{B}_{\ b}\ {}^{\circ}\underline{\mathbf{g}}_{A}\ {}^{\circ}\underline{\mathbf{g}}_{B} \\
&= {}^{t}t^{ab}\ ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X^{-1})^{B}_{\ b}\ {}^{\circ}g_{AL}\ {}^{\circ}g_{BM}\ {}^{\circ}\underline{\mathbf{g}}^{L}\ {}^{\circ}\underline{\mathbf{g}}^{M} \\
&= {}^{t}t^{ab}\ ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X^{-1})^{B}_{\ b}\ {}^{\circ}g_{AL}\ {}^{\circ}\underline{\mathbf{g}}^{L}\ {}^{\circ}\underline{\mathbf{g}}_{B}\ , \qquad (2.87a)
\end{aligned}$$

$$\begin{aligned}
{}^{t}\underline{\underline{\mathbf{T}}}^{\natural} &= {}^{t}t^{a}_{\ b}\ ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X)^{b}_{\ B}\ {}^{\circ}g^{BL}\ {}^{\circ}\underline{\mathbf{g}}_{A}\ {}^{\circ}\underline{\mathbf{g}}_{L} \\
&= {}^{t}t^{a}_{\ b}\ ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X)^{b}_{\ B}\ {}^{\circ}g_{AL}\ {}^{\circ}\underline{\mathbf{g}}^{L}\ {}^{\circ}\underline{\mathbf{g}}^{B} \\
&= {}^{t}t^{a}_{\ b}\ ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X)^{b}_{\ B}\ {}^{\circ}\underline{\mathbf{g}}_{A}\ {}^{\circ}\underline{\mathbf{g}}^{B}\ , \qquad (2.87b)
\end{aligned}$$

$$\begin{aligned}
{}^{t}\underline{\underline{\mathbf{T}}}^{\flat} &= {}^{t}t_{ab}\ ({}^{t}_{\circ}X)^{a}_{\ A}\ ({}^{t}_{\circ}X)^{b}_{\ B}\ {}^{\circ}g^{AL}\ {}^{\circ}g^{BM}\ {}^{\circ}\underline{\mathbf{g}}_{L}\ {}^{\circ}\underline{\mathbf{g}}_{M} \\
&= {}^{t}t_{ab}\ ({}^{t}_{\circ}X)^{a}_{\ A}\ ({}^{t}_{\circ}X)^{b}_{\ B}\ {}^{\circ}\underline{\mathbf{g}}^{A}\ {}^{\circ}\underline{\mathbf{g}}^{B} \\
&= {}^{t}t_{ab}\ ({}^{t}_{\circ}X)^{a}_{\ A}\ ({}^{t}_{\circ}X)^{b}_{\ B}\ {}^{\circ}g^{BL}\ {}^{\circ}\underline{\mathbf{g}}^{A}\ {}^{\circ}\underline{\mathbf{g}}_{L}\ . \qquad (2.87c)
\end{aligned}$$

For two tensors ${}^{t}\underline{\underline{\mathbf{t}}}$ and ${}^{t}\underline{\underline{\mathbf{w}}}$ defined in the spatial configuration at $\chi\,({}^{t}x^{a}, a = 1,2,3)$, using Eqs. (2.84a-2.84c), it is easy to show that:

$$ {}^{t}\underline{\underline{\mathbf{T}}}^{\flat} : {}^{t}\underline{\underline{\mathbf{W}}}^{\sharp} = {}^{t}\underline{\underline{\mathbf{T}}}^{\sharp} : {}^{t}\underline{\underline{\mathbf{W}}}^{\flat} = {}^{t}\underline{\underline{\mathbf{t}}} : {}^{t}\underline{\underline{\mathbf{w}}} \qquad (2.88)$$

and also, using Eqs.(2.87a-2.87c), we can show that

$$\begin{aligned}
[{}^{t}\underline{\underline{\mathbf{T}}}^{\flat}]_{AB} &= [{}^{t}\underline{\underline{\mathbf{T}}}^{\natural}]^{P}_{\ B}\ {}^{t}_{\circ}C_{PA} \qquad (2.89a) \\
[{}^{t}\underline{\underline{\mathbf{T}}}^{\flat}]_{AB} &= [{}^{t}\underline{\underline{\mathbf{T}}}^{\sharp}]^{PQ}\ {}^{t}_{\circ}C_{PA}\ {}^{t}_{\circ}C_{QB}\ . \qquad (2.89b)
\end{aligned}$$

If in the spatial configuration, at the point $\chi\,({}^{t}x^{a}, a = 1,2,3)$, two vectors ${}^{t}\underline{\mathbf{b}}$ and ${}^{t}\underline{\mathbf{w}}$ are related by the second order tensor ${}^{t}\underline{\underline{\mathbf{t}}}$, via the equation,

$$ {}^{t}\underline{\mathbf{b}} = {}^{t}\underline{\underline{\mathbf{t}}} \cdot {}^{t}\underline{\mathbf{w}} \qquad (2.90)$$

in the reference configuration, we can easily verify that the following relations are fulfilled:

$$\begin{aligned}
{}^{t}\underline{\mathbf{B}}^{\sharp} &= {}^{t}\underline{\underline{\mathbf{T}}}^{\sharp} \cdot {}^{t}\underline{\mathbf{W}}^{\flat} \qquad (2.91a) \\
{}^{t}\underline{\mathbf{B}}^{\flat} &= {}^{t}\underline{\underline{\mathbf{T}}}^{\flat} \cdot {}^{t}\underline{\mathbf{W}}^{\sharp} \qquad (2.91b) \\
{}^{t}\underline{\mathbf{B}}^{\sharp} &= {}^{t}\underline{\underline{\mathbf{T}}}^{\natural} \cdot {}^{t}\underline{\mathbf{W}}^{\sharp}\ . \qquad (2.91c)
\end{aligned}$$

Example 2.13. ◀◀◀◀◀

Instead of using the mixed components ${}^{t}t^{i}{}_{j}$, it is possible to use the components ${}^{t}t_{j}{}^{i}$, hence

$$ {}^{t}\underline{\underline{\mathbf{t}}} = {}^{t}t_{j}{}^{i}\ {}^{t}\underline{\mathbf{g}}^{j}\ {}^{t}\underline{\mathbf{g}}_{i} = {}^{t}\tilde{t}_{j}{}^{i}\ {}^{t}\underline{\tilde{\mathbf{g}}}^{j}\ {}^{t}\underline{\tilde{\mathbf{g}}}_{i} $$

we can then define in the reference configuration the second order tensor,

$$ {}^{t}\underline{\underline{\mathbf{T}}}^{\circ} = {}^{t}\tilde{t}_{j}{}^{i}\ {}^{\circ}\underline{\tilde{\mathbf{g}}}^{j}\ {}^{\circ}\underline{\tilde{\mathbf{g}}}_{i} = [{}^{t}\underline{\underline{\mathbf{T}}}^{\circ}]_{A}^{\ B}\ {}^{\circ}\underline{\mathbf{g}}^{A}\ {}^{\circ}\underline{\mathbf{g}}_{B} $$

after some algebra we then get,

$$ [{}^{t}\underline{\underline{\mathbf{T}}}^{\circ}]_{A}^{\ B} = {}^{t}t_{a}{}^{b}\ ({}^{t}_{\circ}X^{-1})^{B}_{\ b}\ ({}^{t}_{\circ}X)^{a}{}_{A} \quad . $$

Therefore,

$$ [{}^{t}\phi^{*}\ ({}^{t}t_{j}{}^{i})]_{A}^{\ B} = [{}^{t}\underline{\underline{\mathbf{T}}}^{\circ}]_{A}^{\ B} \ . $$

Starting from Eq.(2.90), we can also show that,

$$ {}^{t}\underline{\mathbf{B}}^{\flat} = {}^{t}\underline{\underline{\mathbf{T}}}^{\circ} \cdot {}^{t}\underline{\mathbf{W}}^{\flat} \ . $$

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2.10 Tensors in the spatial configuration from representations in the reference configuration ("push-forward")

In the previous section we obtained representations in the reference configuration of tensors defined in the spatial configuration (pull-back).

The inverse operation is named, in the manifold analysis literature (Lang 1972, Marsden & Hughes 1983), *push-forward*. For a second-order tensor, from Eqs.(2.85a-2.85c), we can write:

$$ {}^{t}t^{ab} = \left[\ {}^{t}\phi_{*}[{}^{t}\underline{\underline{\mathbf{T}}}^{\sharp}]^{AB}\right]^{ab} = {}^{t}_{\circ}X^{a}{}_{A}\ {}^{t}_{\circ}X^{b}{}_{B}\ [{}^{t}\underline{\underline{\mathbf{T}}}^{\sharp}]^{AB} \tag{2.92a} $$

$$ {}^{t}t_{ab} = \left[\ {}^{t}\phi_{*}[{}^{t}\underline{\underline{\mathbf{T}}}^{\flat}]_{AB}\right]_{ab} = ({}^{t}_{\circ}X^{-1})^{A}_{\ a}\ ({}^{t}_{\circ}X^{-1})^{B}_{\ b}\ [{}^{t}\underline{\underline{\mathbf{T}}}^{\flat}]_{AB} \tag{2.92b} $$

$$ {}^{t}t^{a}_{\ b} = \left[\ {}^{t}\phi_{*}[{}^{t}\underline{\underline{\mathbf{T}}}^{\natural}]^{A}_{\ B}\right]^{a}_{\ b} = {}^{t}_{\circ}X^{a}{}_{A}\ ({}^{t}_{\circ}X^{-1})^{B}_{\ b}\ [{}^{t}\underline{\underline{\mathbf{T}}}^{\natural}]^{A}_{\ B} \ . \tag{2.92c} $$

Example 2.14. ◀◀◀◀◀

Using Eqs. (2.92a-2.92c) we can show that,

$$\left[{}^t\phi_*(W_A)\right]_a \;=\; ({}^tb^{-1})_{al}\;\left[{}^t\phi_*(W^A)\right]^l \;.$$

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2.11 Pull-back/push-forward relations between strain measures

In the two previous Sections we defined representations of the types known as pull-back and push-forward. As we will see later on, these kinds of representations are extremely useful in nonlinear continuum mechanics.

In Sect. 2.8 we defined a number of strain measures, some of them in the spatial configuration (e.g. the Finger and Almansi tensors) and some of them in the reference configuration (e.g. the Green, Green-Lagrange and Hencky tensors).

In the present section we will establish relations of the pull-back/push-forward type between those strain measures.

We define, as usual, at the material point χ:

∘ In the spatial configuration a coordinate system $\{{}^tx^a\}$ with covariant base vectors ${}^t\underline{\mathbf{g}}_a$.

∘ In the reference configuration a coordinate system $\{{}^\circ x^A\}$ with covariant base vectors ${}^\circ\underline{\mathbf{g}}_A$.

We can therefore calculate at χ the deformation tensor ${}^t_\circ\underline{\underline{\mathbf{X}}}$.

- *The pull-backs of the spatial metric tensor are*:

$$\left[{}^t\phi^*({}^tg_{ab})\right]_{AB}\;{}^\circ\underline{\mathbf{g}}^A\;{}^\circ\underline{\mathbf{g}}^B \;=\; {}^t\underline{\underline{\mathbf{g}}}^\flat \;=\; {}^t_\circ\underline{\underline{\mathbf{C}}}\;, \tag{2.93a}$$

$$\left[{}^t\phi^*({}^tg^{ab})\right]^{AB}\;{}^\circ\underline{\mathbf{g}}_A\;{}^\circ\underline{\mathbf{g}}_B \;=\; {}^t\underline{\underline{\mathbf{g}}}^\sharp \;=\; {}^t_\circ\underline{\underline{\mathbf{C}}}^{-1}\;. \tag{2.93b}$$

- *The pull-back of the Almansi strain tensor is*:

$$\left[{}^t\phi^*({}^te_{ab})\right]_{AB}\;{}^\circ\underline{\mathbf{g}}^A\;\;{}^\circ\underline{\mathbf{g}}^B \;=\; {}^t\underline{\underline{\mathbf{E}}}^\flat \;=\; {}^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}\;. \tag{2.94a}$$

From Eqs. (2.77),(2.80a-2.80c) and (2.91a-2.91c), we get

$${}^t\mathrm{d}\underline{\mathbf{x}}\;\cdot\;{}^t\underline{\underline{\mathbf{e}}}\;\cdot\;{}^t\mathrm{d}\underline{\mathbf{x}} \;=\; {}^\circ\mathrm{d}\underline{\mathbf{x}}\;\cdot\;{}^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}\cdot\;{}^\circ\mathrm{d}\underline{\mathbf{x}}\;. \tag{2.94b}$$

- *The pull-back of the left stretch tensor is*:

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{\natural} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \;. \tag{2.95} $$

- In many problems related to metallurgy (e.g. metal-forming problems) the usual practice is to use a logarithmic strain measure (Hill 1978). Therefore, in the spatial configuration the following strain measure is defined

$$ {}^{t}\underline{\underline{\mathbf{h}}} \;=\; \ln {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \;. \tag{2.96a} $$

The pull-back of the tensor ${}^{t}\underline{\underline{\mathbf{h}}}$ is:

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{H}}} \;=\; {}^{t}\underline{\underline{\mathbf{H}}}^{\natural} \;=\; \ln {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{\natural} \;=\; \ln {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \;. \tag{2.96b} $$

The above tensor is known as the *Hencky deformation tensor.*
In following Chapters we will see the importance of the Hencky strain tensor for the analysis of finite strain problems (Anand 1979). Some recent finite element formulations, developed for the analysis of finite strain elasto-plastic problems use this strain measure (e.g. Rolph & Bathe 1984, Weber & Anand 1990, Eterovic & Bathe 1990, Simo 1991, Dvorkin, Pantuso & Repetto 1992\1993 \1994\1995, Dvorkin 1995a\1995b \1995c, Pèric, Owen & Honnor 1992).
It is useful to note that we also get Eqs. (2.95) and (2.96a-2.96b) when instead of searching for a representation in the reference configuration, we search for a representation in a configuration rotated by ${}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T}$ from the spatial configuration: *corotational representation.*

- *The push-forward of the reference configuration metric tensor is*:

$$ \begin{aligned} \left[\, {}^{t}\phi_{*}(\, {}^{\circ}g_{AB})\,\right]_{ab}\; {}^{t}\underline{\mathbf{g}}^{a}\; {}^{t}\underline{\mathbf{g}}^{b} \;&=\; ({}^{t}_{\circ}X^{-1})^{A}_{\;\;a}\; ({}^{t}_{\circ}X^{-1})^{B}_{\;\;b}\; {}^{\circ}g_{AB}\; {}^{t}\underline{\mathbf{g}}^{a}\; {}^{t}\underline{\mathbf{g}}^{b} \\ &=\; {}^{t}\underline{\underline{\mathbf{b}}}^{-1} \end{aligned} \tag{2.97a} $$

$$ \begin{aligned} \left[\, {}^{t}\phi_{*}(\, {}^{\circ}g^{AB})\,\right]^{ab}\; {}^{t}\underline{\mathbf{g}}_{a}\; {}^{t}\underline{\mathbf{g}}_{b} \;&=\; {}^{t}_{\circ}X^{a}_{\;\;A}\; {}^{t}_{\circ}X^{b}_{\;\;B}\; {}^{\circ}g^{AB}\; {}^{t}\underline{\mathbf{g}}_{a}\; {}^{t}\underline{\mathbf{g}}_{b} \\ &=\; {}^{t}\underline{\underline{\mathbf{b}}} \;. \end{aligned} \tag{2.97b} $$

In Chap. 3 we will study relations of the pull-back/push-forward type between stress measures.

2.12 Objectivity

The description of physical phenomena using *objective formulations* is a central topic in continuum mechanics.

We will first present the classical concept of objectivity under rotations and translations (isometries) (Truesdell & Noll 1965). Next, we will present the concept of objectivity under general changes of the reference frame (Marsden & Hughes 1983), we will use the word *covariance* to refer to this concept.

2.12.1 Reference frame and isometric transformations

We call an *event* the pair $\{{}^t\underline{\mathbf{x}}, t\}$ formed by a vector ${}^t\underline{\mathbf{x}}$ that defines a point in the Euclidean space and a time t.

A *reference or observation frame* is a way of relating the physical world to the points in an $\Re^3$ Euclidean space and a real axis of time (Truesdell & Noll 1965).

Examples of reference frames:

- The system of fixed stars and a clock.
- The walls of my office and my watch.
- A system of coordinates drawn on a rotating platform and a clock.

An *isometric transformation of reference frame* is a mapping $(\Re^3, t) \rightarrow (\Re^3, t)$ in which the distances between spatial points, the time intervals between events and the time ordering of events are preserved. Obviously, isometric transformations restrict us to Newtonian Mechanics.

To describe an isometric transformation of reference frame we define in the spatial configuration:

- A fixed Cartesian system $\{{}^t z_\alpha\}$.
- Another Cartesian system $\{{}^{t^*} z^*_\alpha\}$ that rotates and translates.

For simultaneous events, in the fixed system we register a time t, and in the moving system a time t^*.

The base vectors of the fixed and moving Cartesian frames are ${}^t\underline{\mathbf{e}}_\alpha$ and ${}^{t^*}\underline{\mathbf{e}}^*_\alpha$, respectively; hence, we can write, at an instant t:

$$ {}^t\underline{\mathbf{e}}_\alpha = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t^*}\underline{\mathbf{e}}^*_\alpha \,. \tag{2.98} $$

Obviously, the tensor $\underline{\underline{\mathbf{Q}}}(t)$ is orthogonal.

Let us call $\underline{\mathbf{c}}(t)$ the vector that goes from ${}^t\underline{\mathbf{z}} = (0,0,0)$ (origin of the fixed Cartesian frame) to ${}^{t^*}\underline{\mathbf{z}}^* = (0,0,0)$ (origin of the moving Cartesian frame).

An observer on the moving frame defines an *event* with the vector ${}^{t^*}\underline{\mathbf{z}}^*$ and the time t^*.

An observer on the fixed frame defines the *same event* with the vector ${}^t\underline{\mathbf{z}}$ and the time t.

Let us assume that the two observers are observing a lab experiment, e.g. the measurement of the stretching of a spring when it is loaded with a weight of 1 kg. Let us also assume that the lab is moving with the moving frame; for the observer on this frame ${}^{t^*}\underline{\mathbf{z}}^*$ is the position vector of the spring-end.

The observer on the fixed frame will see the spring-end moving:

$$ {}^t\underline{\mathbf{z}}(t) = \underline{\mathbf{c}}(t) + \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t^*}\underline{\mathbf{z}}^* \,. \tag{2.99a} $$

Since we are considering now isometric transformations, the following condition holds:

$$t = t^* - a \,, \tag{2.99b}$$

where a is a constant. For simplicity, from here on, we will consider $a = 0$ ($t = t^*$).

If we include the possibility that $\underline{\underline{\mathbf{Q}}}(t)$ represents not only rotations but also reflections (transformations between left-handed and right-handed systems), Eqs. (2.99a-2.99b) *represent the most general isometric transformation* (Truesdell & Noll 1965, Truesdell 1966).

We should not confuse the concept of *change of reference frame* with the concept of *change of coordinates system*, the latter being an instant concept does not incorporate the frame velocity.

An isometric change of reference frame may induce changes in scalars, vectors and tensors.

- *Scalars* do not change during an isometric transformation.
- *Vectors* defined in the spatial configuration can always be considered as proportional to the difference between two position vectors in that configuration (Truesdell & Noll 1965); hence, using Eq. (2.99a) we get

$$^{t}\underline{\mathbf{v}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}\underline{\mathbf{v}}^* \,. \tag{2.100}$$

  The above is the transformation law for spatial vectors under an isometric transformation of reference frame.
- *Second-order tensors* defined in the spatial configuration relate vectors defined in the same configuration (quotient rule)[6]. In the moving frame, we can write for an arbitrary second-order tensor ${}^{t}\underline{\underline{\mathbf{s}}}^*$

$$^{t}\underline{\mathbf{v}}^* = {}^{t}\underline{\underline{\mathbf{s}}}^* \cdot {}^{t}\underline{\mathbf{w}}^* \,. \tag{2.101a}$$

  Using Eq.(2.100),

$$^{t}\underline{\mathbf{v}} = \left[\underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}\underline{\underline{\mathbf{s}}}^* \cdot \underline{\underline{\mathbf{Q}}}^T(t)\right] \cdot {}^{t}\underline{\mathbf{w}} \,. \tag{2.101b}$$

  Therefore, the transformation law for spatial second-order tensors under an isometric transformation of reference frame is

$$^{t}\underline{\underline{\mathbf{s}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}\underline{\underline{\mathbf{s}}}^* \cdot \underline{\underline{\mathbf{Q}}}^T(t) \,. \tag{2.101c}$$

- For spatial tensors of higher order, the transformation laws can be derived in an identical way.

[6] See Appendix.

2.12.2 Objectivity or material-frame indifference

Let us assume a deformation process taking place in the moving reference frame. This deformation process can be referred to a fixed reference configuration.

There are general tensors (including scalars and vectors) defined in the reference configuration (e.g. the Green-Lagrange strain tensor), they are called *material or Lagrangian tensors.*

There are general tensors defined in the spatial configuration (e.g. the Almansi strain tensor, the material velocity), they are called *spatial or Eulerian tensors.*

There are general tensors defined in both configurations (e.g. the deformation gradient tensor), they are called *two-point tensors.*

Following (Lubliner 1985) we define the following *objectivity criteria* under isometric transformations of a reference frame (*classical objectivity*):

- A *Lagrangian tensor is objective* if it is not affected by changes of the reference frame.
- An *Eulerian tensor is objective* if, under a change of reference frame, transforms according to Eqs. (2.100) and (2.101c).
- A *two points second order tensor is objective* if, when operating on a Eulerian objective vector, produces a Lagrangian objective vector.

Example 2.15. ◀◀◀◀◀

If we differentiate Eq. (2.99a) with respect to time we get,

$$ {}^{t}\underline{\dot{\mathbf{z}}}(t) = {}^{t}\underline{\dot{\mathbf{c}}}(t) + \underline{\underline{\dot{\mathbf{Q}}}}(t) \cdot {}^{t}\underline{\mathbf{z}}^{\star} + \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}\underline{\dot{\mathbf{z}}}^{\star} \ . $$

Hence, the velocity is not an objective vector. ◀◀◀◀◀

In the same way we can show that the *acceleration vector is not objective.* Therefore, even though forces are spatial objective tensors, Newton's second law (${}^{t}\underline{\mathbf{F}} = m\ {}^{t}\underline{\ddot{\mathbf{x}}}$) is not objective and is only applicable in *inertial frames* (Truesdell & Noll 1965).

In what follows, we will analyze the objectivity of some of the tensors previously defined:

- The *deformation gradient tensor*, in the moving frame, satisfies the following relation:

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}^{*} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{*} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}} \ . \tag{2.102a} $$

- The vector ${}^{\circ}\mathrm{d}\underline{\mathbf{x}}$ is an objective Lagrangian vector and ${}^{t}\mathrm{d}\underline{\mathbf{x}}^*$ is an objective Eulerian vector, hence ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^*$ fulfills the objectivity definition.
  In the fixed frame,

$$\underline{\underline{\mathbf{Q}}}^T(t) \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^* \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}} \, , \tag{2.102b}$$

  and finally,

$${}^{t}_{\circ}\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^* \, . \tag{2.102c}$$

  Equation (2.102c) is the transformation law for *objective two-point second-order tensors.*
- Performing a right polar decomposition on both sides of Eq. (2.102c), we get

$${}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^* \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}}^* \, . \tag{2.103a}$$

  Taking into account that:
  - ◦ The inner (dot) product of two orthogonal second-order tensors is an orthogonal second order tensor.
  - ◦ The polar decomposition is unique.

  We get

$${}^{t}_{\circ}\underline{\underline{\mathbf{R}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^* \tag{2.103b}$$

$${}^{t}_{\circ}\underline{\underline{\mathbf{U}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{U}}}^* \, . \tag{2.103c}$$

  Hence, the *rotation tensor* (two-point tensor) and the *right stretch tensor* (Lagrangian tensor) are *objective* under isometric transformations.
- From Eq. (2.47b), we can write

$${}^{t}_{\circ}\underline{\underline{\mathbf{V}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^T \, . \tag{2.104a}$$

  and using Eqs. (2.103a-2.103c), we obtain

$${}^{t}_{\circ}\underline{\underline{\mathbf{V}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^* \cdot \underline{\underline{\mathbf{Q}}}^T(t) \, . \tag{2.104b}$$

  The above equation shows that the *left stretch tensor* (Eulerian tensor) is *objective* under isometric transformations.
- The *Green-Lagrange strain tensor* (Lagrangian tensor) is (Eq. (2.65))

$$ {}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\left[{}^t_{\circ}\underline{\underline{\mathbf{X}}}^T \cdot {}^t_{\circ}\underline{\underline{\mathbf{X}}} - {}^{\circ}\underline{\underline{\mathbf{g}}} \right] \tag{2.105a} $$

and using Eq. (2.102c), we get

$$ {}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} = {}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}}^{\star} \tag{2.105b} $$

The above equation depicts an objective behavior under isometric transformations.

- The *Almansi strain tensor* (Eulerian tensor) is (Eq. (2.68)):

$$ {}^t\underline{\underline{\mathbf{e}}} = \frac{1}{2}\left[{}^t\underline{\underline{\mathbf{g}}} - {}^t_{\circ}\underline{\underline{\mathbf{X}}}^{-T} \cdot {}^t_{\circ}\underline{\underline{\mathbf{X}}}^{-1} \right] . \tag{2.106a} $$

and using Eq. (2.102c), we get

$$ {}^t\underline{\underline{\mathbf{e}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^t\underline{\underline{\mathbf{e}}}^{*} \cdot \underline{\underline{\mathbf{Q}}}^T(t) . \tag{2.106b} $$

The above equation depicts an *objective* behavior under isometric transformations.

2.12.3 Covariance

Let us assume a body that remains undeformed, referred to a spatial coordinate system that keeps changing (e.g. at a time t we have the spatial system $\{{}^t x^a\}$ and at a time $\hat{t}$ the system $\{{}^{\hat{t}}\hat{x}^a\}$). For a spatial second order tensor ${}^t\underline{\underline{\mathbf{a}}}$, using the usual tensor transformations, we can write (Truesdell & Noll 1965):

$$ {}^{\hat{t}}\hat{a}^{ab} = \frac{\partial^{\hat{t}}\hat{x}^a}{\partial^t x^l}\,\frac{\partial^{\hat{t}}\hat{x}^b}{\partial^t x^m}\;{}^t a^{lm} \,, \tag{2.107a} $$

$$ {}^{\hat{t}}\hat{a}_{ab} = \frac{\partial^t x^l}{\partial^{\hat{t}}\hat{x}^a}\,\frac{\partial^t x^m}{\partial^{\hat{t}}\hat{x}^b}\;{}^t a_{lm} \,, \tag{2.107b} $$

$$ {}^{\hat{t}}\hat{a}^{a}{}_{b} = \frac{\partial^{\hat{t}}\hat{x}^a}{\partial^t x^l}\,\frac{\partial^t x^m}{\partial^{\hat{t}}\hat{x}^b}\;{}^t a^{l}{}_{m} \,. \tag{2.107c} $$

Between the spatial coordinates of the material particles at t and the spatial coordinates of the same particles at $\hat{t}$ we can define a mapping $({}^{\hat{t}}_{t}\phi)$ (see Eq. (2.4)). Hence, we can also define a deformation gradient tensor, ${}^{\hat{t}}_{t}\underline{\underline{\mathbf{X}}}$, and we can rewrite Eqs. (2.107a-2.107c) as

$$ {}^{t}a^{ab} = ({}^{\hat{t}}_{t}X^{-1})^{a}_{\ l} \, ({}^{\hat{t}}_{t}X^{-1})^{b}_{\ m} \; {}^{\hat{t}}\hat{a}^{lm} = \left[{}^{\hat{t}}_{t}\phi^{*}({}^{\hat{t}}\hat{a}^{lm}\,)\right]^{ab} , \qquad (2.108a) $$

$$ {}^{t}a_{ab} = {}^{\hat{t}}_{t}X^{l}_{\ a} \; {}^{\hat{t}}_{t}X^{m}_{\ b} \; {}^{\hat{t}}\hat{a}_{lm} = \left[{}^{\hat{t}}_{t}\phi^{*}({}^{\hat{t}}\hat{a}_{lm}\,)\right]_{ab} , \qquad (2.108b) $$

$$ {}^{t}a^{a}_{\ b} = ({}^{\hat{t}}_{t}X^{-1})^{a}_{\ l} \; {}^{\hat{t}}_{t}X^{m}_{\ b} \; {}^{\hat{t}}\hat{a}^{l}_{\ m} = \left[{}^{\hat{t}}_{t}\phi^{*}({}^{\hat{t}}\hat{a}^{l}_{\ m}\,)\right]^{a}_{\ b} . \qquad (2.108c) $$

If we consider now general transformations between the spatial configurations at t and $\hat{t}$ (not necessarily restricted to simple changes of coordinate system) the *spatial tensor* ${}^{t}\underline{\underline{\mathbf{a}}}$ is defined as *covariant* or *objective* (in a more general way than the above defined objectivity under isometric transformations) if under the mapping $({}^{\hat{t}}_{t}\phi)$ it transforms following Eqs.(2.108a-2.108c) (Marsden & Hughes 1983).

For a *two-point tensor*, the covariance criterion is only applied to the indices associated to the spatial basis (Lubliner 1985).

A *Lagrangian tensor* is always covariant if it is not affected by changes of the reference frame. The reader can easily check that for the case of isometric transformations the concept of covariance is coincident with the classical concept of objectivity presented in the previous section.

> A physical law is objective (either in the classical or in the covariant sense) if all the tensors in its mathematical formulation are objective.

2.13 Strain rates

In the previous sections we presented a static description of the kinematics of continuous media: given a spatial and a reference configuration we developed tools to relate both configurations (deformation gradient tensor, strain measures, etc.)

In the present section we will study the *kinematic evolution* of the spatial configuration. For this purpose we will introduce the *time rates* of the different tensors described above.

2.13.1 The velocity gradient tensor

The time rate of Eq. (2.23) is

$$ {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}} = \left[\frac{\partial^{t}v^{a}}{\partial^{\circ}x^{A}} + {}^{t}_{\circ}X^{l}_{\ A} \; {}^{t}\Gamma^{a}_{lb} \; {}^{t}v^{b}\right] \; {}^{t}\underline{\mathbf{g}}_{a} \; {}^{\circ}\underline{\mathbf{g}}^{A} , \qquad (2.109a) $$

where ${}^{t}v^{a}$ was defined in Eq. (2.10).

It is important to remember that the functional dependence is

$$ {}_{\circ}^{t}\dot{X}^{a}{}_{A} \;=\; {}_{\circ}^{t}\dot{X}^{a}{}_{A}({}^{\circ}x^{B}, t) \; . \tag{2.109b} $$

Using the chain rule in Eq. (2.109a),

$$ {}_{\circ}^{t}\dot{X}^{a}{}_{A} \;=\; {}^{t}v^{a}|_{l}\; {}_{\circ}^{t}X^{l}{}_{A} \; . \tag{2.109c} $$

We define in the spatial configuration the *velocity gradient tensor*,

$$ {}^{t}\underline{\underline{\mathbf{l}}} \;=\; {}^{t}v^{a}|_{l}\; {}^{t}\underline{\mathbf{g}}_{a}\; {}^{t}\underline{\mathbf{g}}^{l} \; , \tag{2.110a} $$

we can write the above as

$$ {}^{t}\underline{\underline{\mathbf{l}}} \;=\; {}^{t}\underline{\mathbf{v}}\; \underline{\boldsymbol{\nabla}} \; , \tag{2.110b} $$

$$ {}^{t}\underline{\underline{\mathbf{l}}}^{T} \;=\; \underline{\boldsymbol{\nabla}}\, {}^{t}\underline{\mathbf{v}} \; . \tag{2.110c} $$

Hence, we can write Eq.(2.109c) as

$$ {}_{\circ}^{t}\dot{X}^{a}{}_{A} \;=\; {}^{t}l^{a}{}_{l}\; {}_{\circ}^{t}X^{l}{}_{A} \tag{2.111a} $$

and therefore,

$$ {}_{\circ}^{t}\dot{\underline{\underline{\mathbf{X}}}} = {}^{t}\underline{\underline{\mathbf{l}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} \; . \tag{2.111b} $$

It is important to realize that the above is the material time derivate of the deformation gradient tensor, ${}_{\circ}^{t}\dot{\underline{\underline{\mathbf{X}}}} = \dfrac{D\, {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}}{Dt}$.

2.13.2 The Eulerian strain rate tensor and the spin (vorticity) tensor

We can decompose the velocity gradient tensor into its symmetric and skew-symmetric components:

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}^{t}\underline{\underline{\mathbf{d}}} + {}^{t}\underline{\underline{\boldsymbol{\omega}}} \tag{2.112a} $$

where,

$$ {}^{t}\underline{\underline{\mathbf{d}}} \;=\; {}^{t}\underline{\underline{\mathbf{d}}}^{T} \;=\; \frac{1}{2}\left({}^{t}\underline{\underline{\mathbf{l}}} \;+\; {}^{t}\underline{\underline{\mathbf{l}}}^{T}\right) \tag{2.112b} $$

is the *Eulerian strain rate tensor* (defined in the spatial configuration) and,

$$ {}^{t}\underline{\underline{\boldsymbol{\omega}}} \;=-\; {}^{t}\underline{\underline{\boldsymbol{\omega}}}^{T} \;=\; \frac{1}{2}\left({}^{t}\underline{\underline{\mathbf{l}}} \;-\; {}^{t}\underline{\underline{\mathbf{l}}}^{T}\right) \tag{2.112c} $$

is the *spin or vorticity tensor*, also defined in the spatial configuration.

Let us assume a deformation process referred to a fixed Cartesian system. The principal directions of ${}_{\circ}^{t}\underline{\underline{\mathbf{U}}}$ form, in the reference configuration, a Cartesian system known as *Lagrangian system*. The principal directions of ${}_{\circ}^{t}\underline{\underline{\mathbf{V}}}$ form, in the spatial configuration, a Cartesian system known as a *Eulerian system* (Hill 1978).

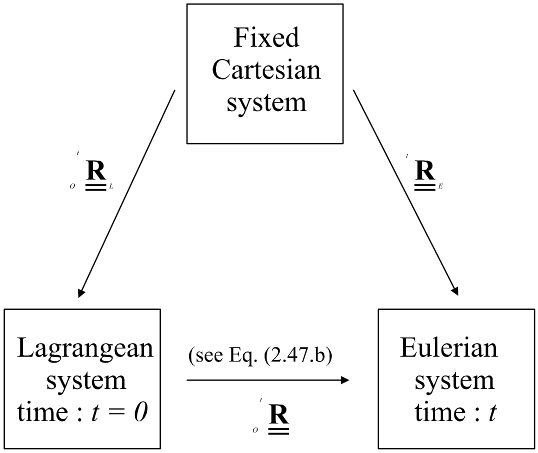

Fig. 2.5. Rotations

We can go from one of the above-defined coordinate systems to another one using the rotation tensors sketched in Fig. 2.5.

From Fig. 2.5, we get

$$ {}^t\underline{\underline{\mathbf{R}}}_E \;=\; {}^t_\circ\underline{\underline{\mathbf{R}}} \;\cdot\; {}^t_\circ\underline{\underline{\mathbf{R}}}_L \;. \tag{2.113} $$

For two consecutive rotations,

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{R}}} \;=\; {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \;\cdot\; {}^t_\circ\underline{\underline{\mathbf{R}}} \tag{2.114a} $$

and therefore,

$$ {}^t_\circ\dot{\underline{\underline{\mathbf{R}}}} = \;\lim \Delta t \rightarrow 0 \left[\frac{{}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} - \;{}^t\underline{\underline{\mathbf{g}}}}{\Delta t} \right] \cdot \; {}^t_\circ\underline{\underline{\mathbf{R}}} \;. \tag{2.114b} $$

We can define a rotation rate

$$ {}^t\underline{\underline{\mathbf{\Omega}}}_R = \;\lim \Delta t \rightarrow 0 \left[\frac{{}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} - \;{}^t\underline{\underline{\mathbf{g}}}}{\Delta t} \right] \tag{2.114c} $$

and using it in Eq. (2.114b) (Hill 1978), we get

$$ {}^t_\circ\dot{\underline{\underline{\mathbf{R}}}} = {}^t\underline{\underline{\mathbf{\Omega}}}_R \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} \tag{2.115a} $$

in the same way,

$$ {}^t_\circ\dot{\underline{\underline{\mathbf{R}}}}_L = {}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}_L \;, \tag{2.115b} $$

$$ {}^t\dot{\underline{\underline{\mathbf{R}}}}_E = {}^t\underline{\underline{\mathbf{\Omega}}}_E \cdot {}^t\underline{\underline{\mathbf{R}}}_E \;. \tag{2.115c} $$

Since the rotation tensors are orthogonal we can write

$$ {}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} = {}^\circ\underline{\underline{\mathbf{g}}} \, , \tag{2.116a} $$

taking the time derivative of the above equation and using Eq. (2.115a), we have

$$ {}^t\underline{\underline{\mathbf{\Omega}}}_R + {}^t\underline{\underline{\mathbf{\Omega}}}_R^T = \underline{\underline{\mathbf{0}}} \tag{2.116b} $$

in the same way,

$$ {}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L + {}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L^T = \underline{\underline{\mathbf{0}}} \, , \tag{2.116c} $$

$$ {}^t\underline{\underline{\mathbf{\Omega}}}_E + {}^t\underline{\underline{\mathbf{\Omega}}}_E^T = \underline{\underline{\mathbf{0}}} \, . \tag{2.116d} $$

The above equations indicate that ${}^t\underline{\underline{\mathbf{\Omega}}}_R$, ${}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L$ and ${}^t\underline{\underline{\mathbf{\Omega}}}_E$ are *skew-symmetric* tensors.

2.13.3 Relations between different rate tensors

The time derivative of Eq. (2.113) leads to

$$ {}^t\underline{\underline{\mathbf{\Omega}}}_E \cdot {}^t\underline{\underline{\mathbf{R}}}_E = {}^t\underline{\underline{\mathbf{\Omega}}}_R \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}_L + {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}_L \, , \tag{2.117a} $$

and therefore,

$$ {}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot ({}^t\underline{\underline{\mathbf{\Omega}}}_E - {}^t\underline{\underline{\mathbf{\Omega}}}_R) \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} = {}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L \, . \tag{2.117b} $$

Using Eqs. (2.111b) and (2.40),

$$ {}^t\underline{\underline{\mathbf{l}}} = {}^t_\circ\underline{\underline{\dot{\mathbf{R}}}} \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}^T + {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\dot{\mathbf{U}}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}^{-1} \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}^T \, , \tag{2.118a} $$

splitting the above equation into its symmetric and skew-symmetric components, we get

$$ {}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot {}^t\underline{\underline{\mathbf{d}}} \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} = \frac{1}{2} \left({}^t_\circ\underline{\underline{\dot{\mathbf{U}}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}^{-1} + {}^t_\circ\underline{\underline{\mathbf{U}}}^{-1} \cdot {}^t_\circ\underline{\underline{\dot{\mathbf{U}}}} \right) \tag{2.118b} $$

and

$$ {}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot ({}^t\underline{\underline{\boldsymbol{\omega}}} - {}^t\underline{\underline{\mathbf{\Omega}}}_R) \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} = \frac{1}{2} \left({}^t_\circ\underline{\underline{\dot{\mathbf{U}}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}^{-1} - {}^t_\circ\underline{\underline{\mathbf{U}}}^{-1} \cdot {}^t_\circ\underline{\underline{\dot{\mathbf{U}}}} \right) . \tag{2.118c} $$

It is very important to recognize that (Hill 1978):

$$ {}^t_\circ\underline{\underline{\mathbf{U}}} = {}^\circ\underline{\underline{\mathbf{g}}} \Longrightarrow {}^t\underline{\underline{\boldsymbol{\omega}}} = {}^t\underline{\underline{\mathbf{\Omega}}}_R \, . \tag{2.118d} $$

An example of the above situation is the beginning of the deformation process ($t = 0$).

In the deformation process depicted in Fig. 2.6 (Truesdell & Noll 1965, Malvern 1969) we can write, using Eqs. (2.30a-2.30b):

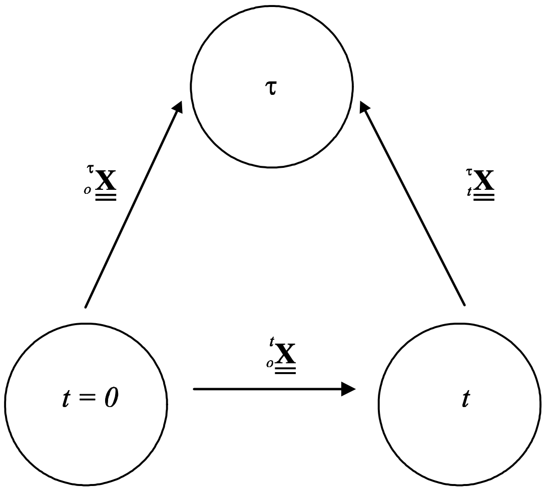

Fig. 2.6. Relative deformation gradients

$$ {}^{\tau}_{\circ}\underline{\underline{\mathbf{X}}} \;=\; {}^{\tau}_{t}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \,. \tag{2.119a} $$

For a fixed t-configuration, we can write

$$ \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{\circ}\underline{\underline{\mathbf{X}}} \;=\; \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \,. \tag{2.119b} $$

Using Eq. (2.111b) and the polar decomposition in the above equation, we get

$$ {}^{\tau}\underline{\underline{\mathbf{l}}} \cdot {}^{\tau}_{\circ}\underline{\underline{\mathbf{X}}} \;=\; \Big(\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{\tau}_{t}\underline{\underline{\mathbf{U}}} \;+\; {}^{\tau}_{t}\underline{\underline{\mathbf{R}}} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{U}}}\Big) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \,, \tag{2.119c} $$

for $\tau = t$ it is obvious that ${}^{\tau}_{t}\underline{\underline{\mathbf{U}}} = {}^{\tau}_{t}\underline{\underline{\mathbf{R}}} = {}^{\circ}\underline{\underline{\mathbf{g}}}$, and since the above equation has to hold for *any* ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}}$,

$$ \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{R}}}\,\Big|_{\tau=t} \;+\; \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{U}}}\,\Big|_{\tau=t} \;=\; {}^{t}\underline{\underline{\mathbf{l}}} \,. \tag{2.119d} $$

It is easy to show that the first tensor on the l.h.s. of the above equation is skew-symmetric and the second one is symmetric; hence,

$$ \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{R}}}\,\Big|_{\tau=t} \;=\; \frac{1}{2}({}^{t}\underline{\underline{\mathbf{l}}} \;-\; {}^{t}\underline{\underline{\mathbf{l}}}^{T}) \;=\; {}^{t}\underline{\underline{\boldsymbol{\omega}}} \tag{2.120a} $$

$$ \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\tau}_{t}\underline{\underline{\mathbf{U}}}\,\Big|_{\tau=t} \;=\; \frac{1}{2}({}^{t}\underline{\underline{\mathbf{l}}} \;+\; {}^{t}\underline{\underline{\mathbf{l}}}^{T}) \;=\; {}^{t}\underline{\underline{\mathbf{d}}} \,. \tag{2.120b} $$

We can obtain an interesting picture of the deformation process by referring the Lagrangian tensors to the Lagrangian coordinate system (principal directions of ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$) and the Eulerian tensors to the Eulerian coordinate system (principal directions of ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$). Hence (Hill 1978):

- In the *Lagrangian system* the components of ${}^t_\circ\underline{\underline{\mathbf{U}}}$ are ${}^t\lambda_r$ (we assume them to be different); the components of ${}^t_\circ\dot{\underline{\underline{\mathbf{U}}}}$ are ${}^t\dot{\lambda}_{rs}$ and the components of ${}^t_\circ\underline{\underline{\mathbf{\Omega}}}_L$ are ${}^t_\circ\Omega^L_{rs}$.
- In the *Eulerian system* the components of ${}^t_\circ\underline{\underline{\mathbf{V}}}$ are of course also ${}^t\lambda_r$; the components of ${}^t\underline{\underline{\mathbf{d}}}$ are ${}^td_{rs}$; the components of ${}^t\underline{\underline{\boldsymbol{\omega}}}$ are ${}^t\omega_{rs}$; the components of ${}^t\underline{\underline{\mathbf{\Omega}}}_E$ are ${}^t\Omega^E_{rs}$ and the components of ${}^t\underline{\underline{\mathbf{\Omega}}}_R$ are ${}^t\Omega^R_{rs}$.

From Eq. (2.117b) we have

$$ {}^t\Omega^E_{rs} \; - \; {}^t\Omega^R_{rs} \; = \; {}^t_\circ\Omega^L_{rs} \; , \tag{2.121a} $$

from Eq. (2.118b) we get

$$ {}^td_{rs} \; = \; {}^t\dot{\lambda}_{rs} \, \frac{{}^t\lambda_r \; + \; {}^t\lambda_s}{2\,{}^t\lambda_r\,{}^t\lambda_s} \tag{2.121b} $$

and from Eq. (2.118c) we get

$$ {}^t\omega_{rs} \; - \; {}^t\Omega^R_{rs} \; = \; {}^t\dot{\lambda}_{rs} \, \frac{{}^t\lambda_r \; - \; {}^t\lambda_s}{2\,{}^t\lambda_r\,{}^t\lambda_s} \; ; \tag{2.121c} $$

(in Eqs. (2.121b) and (2.121c) we do not use the summation convention).

In the fixed Cartesian system the components of ${}^t_\circ\underline{\underline{\mathbf{U}}}$ form the matrix $[{}^t_\circ U]$; hence,

$$ [{}^t_\circ U] \; = \; [{}^t_\circ R_L] \, [{}^t\Lambda] \, [{}^t_\circ R_L]^T \; , \tag{2.122a} $$

where,

$$ [{}^t\Lambda] \; = \; \begin{bmatrix} {}^t\lambda_1 & 0 & 0 \\ 0 & {}^t\lambda_2 & 0 \\ 0 & 0 & {}^t\lambda_3 \end{bmatrix} \; . \tag{2.122b} $$

Taking the time derivative of Eq. (2.122a), we obtain

$$ \begin{aligned} [{}^t_\circ \dot{U}] \; = \; & [{}^t_\circ R_L] \, [{}^t\dot{\Lambda}] \, [{}^t_\circ R_L]^T \; + \; [{}^t_\circ \Omega_L] \, [{}^t_\circ R_L] \, [{}^t\Lambda] \, [{}^t_\circ R_L]^T \\ & - \; [{}^t_\circ R_L] \, [{}^t\Lambda] \, [{}^t_\circ R_L]^T \, [{}^t_\circ \Omega_L] \end{aligned} \tag{2.122c} $$

hence,

$$ \begin{aligned} [{}^t_\circ R_L]^T \, [{}^t_\circ \dot{U}] \, [{}^t_\circ R_L] \; = \; & [{}^t\dot{\Lambda}] \; + \; [{}^t_\circ R_L]^T \, [{}^t_\circ \Omega_L] \, [{}^t_\circ R_L] \, [{}^t\Lambda] \\ & - \; [{}^t\Lambda] \, [{}^t_\circ R_L]^T \, [{}^t_\circ \Omega_L][{}^t_\circ R_L] \; . \end{aligned} \tag{2.122d} $$

The above equation shows that:

$$ \begin{aligned} {}^t\dot{\lambda}_{rs} &= {}^t\dot{\lambda}_r & \qquad (r = s) \\ {}^t\dot{\lambda}_{rs} &= (\, {}^t\lambda_s - \, {}^t\lambda_r) {}^t_\circ\Omega^L_{rs} & \qquad (r \neq s) \end{aligned} \; . \tag{2.123} $$

From Eq. (2.121b) for the case $r = s$ (diagonal components), we get

$$ {}^td_{rr} \; = \; \frac{{}^t\dot{\lambda}_r}{{}^t\lambda_r} \; = \; \frac{\mathrm{d}}{\mathrm{d}t} \, (\ln {}^t\lambda_r) \; . \tag{2.124} $$

Example 2.16. ◀◀◀◀◀

Using Eqs. (2.65), (2.35), (2.111b) and (2.112b) we can show that:

$$ {}^t_\circ\dot{\underline{\underline{\varepsilon}}} = \frac{1}{2}\,{}^t_\circ\dot{\underline{\underline{\mathbf{C}}}} = {}^t_\circ\underline{\underline{\mathbf{X}}}^T \cdot \ {}^t\underline{\underline{\mathbf{d}}} \cdot \ {}^t_\circ\underline{\underline{\mathbf{X}}} \ . $$

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Example 2.17. ◀◀◀◀◀

The Hencky strain tensor components in the fixed Cartesian system are,

$$ [{}^t_\circ H] \ = \ [{}^t_\circ R_L] \ \ln\left[{}^t\Lambda\right] \ [{}^t_\circ R_L]^T \ , $$

hence,

$$ \begin{aligned} [{}^t_\circ \dot{H}] \ = \ & [{}^t_\circ R_L] \ [{}^t\Lambda]^{-1} \ [{}^t\dot{\Lambda}] \ [{}^t_\circ R_L]^T \ + [{}^t_\circ \Omega_L] \ [{}^t_\circ R_L] \ \ln\left[{}^t\Lambda\right] \ [{}^t_\circ R_L]^T \\ & - [{}^t_\circ R_L] \ \ln\left[{}^t\Lambda\right] \ [{}^t_\circ R_L]^T \ [{}^t_\circ \Omega_L] \end{aligned} \ . $$

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2.14 The Lie derivative

In the deformation process represented in Fig. 2.2 we can define, for a Eulerian tensor ${}^t\mathbf{t}$, its *Lie derivative* associated to the flow of the spatial configuration (Simo 1988, Marsden & Hughes 1983):

$$ \mathbf{L}_{{}^t\underline{\mathbf{v}}}({}^t\mathbf{t}) \ = \ {}^t\phi_* \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[{}^t\phi^*({}^t\mathbf{t}) \right] \right\} \ . \tag{2.125} $$

As we already know (see Sect. 2.9) the operation of pull-back is not a tensor operation since it operates on components. Hence, for calculating a Lie derivative using Eq. (2.125) it is important to identify the components of ${}^t\mathbf{t}$ that we are using.

The Lie derivative of a scalar is

$$ \mathbf{L}_{{}^t\underline{\mathbf{v}}}\alpha \ = \ \frac{\mathrm{d}}{\mathrm{d}t}\alpha \ = \ \frac{\partial\alpha}{\partial t} \ + \ \frac{\partial\alpha}{\partial\, {}^t x^a} \ {}^t v^a \ . \tag{2.126} $$

The covariant components of the Lie derivative of a spatial vector ${}^t\underline{\mathbf{w}}$ are

$$ \left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\mathbf{w}}\right)_i \ = \ ({}^t_\circ X^{-1})^A_{\ i} \ \left\{ \frac{\mathrm{d}}{\mathrm{d}t}[({}^t_\circ X)^j_{\ A} \ {}^t w_j] \right\} \ , \tag{2.127a} $$

after some algebra,

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\mathbf{w}}\right)_i = \frac{\partial^t w_i}{\partial t} + \frac{\partial^t w_i}{\partial^t x^a}\,{}^t v^a + \frac{\partial^t v^a}{\partial^t x^i}\,{}^t w_a\,. \tag{2.127b}$$

Since,

$$ {}^t w^i\,{}^t w_i = {}^t\alpha \tag{2.128a}$$

we can write

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\mathbf{w}}\right)^i\,{}^t w_i + {}^t w^i\,\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\mathbf{w}}\right)_i = \left(\frac{\mathrm{d}}{\mathrm{d}t}{}^t w^i\right)\,{}^t w_i + {}^t w^i\left(\frac{\mathrm{d}}{\mathrm{d}t}{}^t w_i\right) \tag{2.128b}$$

and from the above we get the contravariant components of the Lie derivative of a spatial vector ${}^t\underline{\mathbf{w}}$,

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\mathbf{w}}\right)^i = \frac{\partial^t w^i}{\partial t} + \frac{\partial^t w^i}{\partial^t x^a}\,{}^t v^a - {}^t w^a\,\frac{\partial^t v^i}{\partial^t x^a}\,. \tag{2.128c}$$

Following the above procedure we can show that the mixed components of the Lie derivative of a general Eulerian tensor ${}^t\mathbf{t}$ are

$$\begin{aligned}\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}({}^t\mathbf{t})\right)^{a\ldots b}{}_{c\ldots d} = {}& \frac{\partial\,{}^t t^{a\ldots b}{}_{c\ldots d}}{\partial t} + \frac{\partial\,{}^t t^{a\ldots b}{}_{c\ldots d}}{\partial^t x^p}\,{}^t v^p \\ & -\frac{\partial\,{}^t v^a}{\partial^t x^p}\,{}^t t^{p\ldots b}{}_{c\ldots d} - \cdots - \frac{\partial\,{}^t v^b}{\partial^t x^p}\,{}^t t^{a\ldots p}{}_{c\ldots d} \\ & +\frac{\partial\,{}^t v^p}{\partial^t x^c}\,{}^t t^{a\ldots b}{}_{p\ldots d} + \cdots + \frac{\partial\,{}^t v^p}{\partial^t x^d}\,{}^t t^{a\ldots b}{}_{c\ldots p}\end{aligned} \tag{2.129}$$

Example 2.18. ◀◀◀◀◀

To calculate the Lie derivative of the spatial metric tensor ${}^t\underline{\underline{\mathbf{g}}}$ we can directly use Eq. (2.125)

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{g}}}\right)_{ij} = {}^t\phi_*\left[\frac{\mathrm{d}}{\mathrm{d}t}\,\left[{}^t\phi^*({}^t\underline{\underline{\mathbf{g}}})\right]_{IJ}\right]\,,$$

using now Eq. (2.93a), we get

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{g}}}\right)_{ij} = \left[{}^t\phi_*({}^t_\circ\dot{\underline{\underline{\mathbf{C}}}})\right]_{ij}\,.$$

Using the result in Example 2.16, we get

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{g}}}\right)_{ij} = \left(2\,{}^t\underline{\underline{\mathbf{d}}}\right)_{ij}\,.$$

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Example 2.19. ◀◀◀◀◀

To calculate the Lie derivative of the Almansi deformation tensor we use Eq. (2.125) and get

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{e}}}\right)_{ij} \;=\; {}^t\phi_* \left[\frac{\mathrm{d}}{\mathrm{d}t}\;\left[{}^t\phi^*({}^t\underline{\underline{\mathbf{e}}})\right]_{IJ}\right]$$

and resorting to Eq.(2.94a),

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{e}}}\right)_{ij} \;=\; \left[{}^t\phi_*({}^t_\circ\dot{\underline{\underline{\boldsymbol{\varepsilon}}}})\right]_{ij} \;.$$

Taking into account the result obtained in Example 2.16 we can finally write

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{e}}}\right)_{ij} \;=\; \left({}^t\underline{\underline{\mathbf{d}}}\right)_{ij} \;.$$

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Example 2.20. ◀◀◀◀◀

The Lie derivative of the Finger deformation tensor is

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{b}}}\right)^{ij} \;=\; {}^t\phi_* \left[\frac{\mathrm{d}}{\mathrm{d}t}\;\left[{}^t\phi^*({}^t\underline{\underline{\mathbf{b}}})\right]^{IJ}\right] \;.$$

Using Eq. (2.97b) we get

$$\left[{}^t\phi^*({}^t\underline{\underline{\mathbf{b}}})\right]^{IJ} \;=\; \left[{}^t\underline{\underline{\mathbf{B}}}^\sharp\right]^{IJ} \;=\; {}^\circ g^{IJ}$$

and since

$${}^\circ\dot{\underline{\underline{\mathbf{g}}}} \;=\; \underline{\underline{\mathbf{0}}}$$

we get

$$\left(\mathbf{L}_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\mathbf{b}}}\right)^{ij} \;=\; 0 \;.$$

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2.14.1 Objective rates and Lie derivatives

In this Section we will show that the Lie derivative is the adequate mathematical tool for deriving covariant (objective) rates from covariant (objective) Eulerian tensors.

Let us consider the deformation processes schematized in Fig. 2.7. It is obvious that

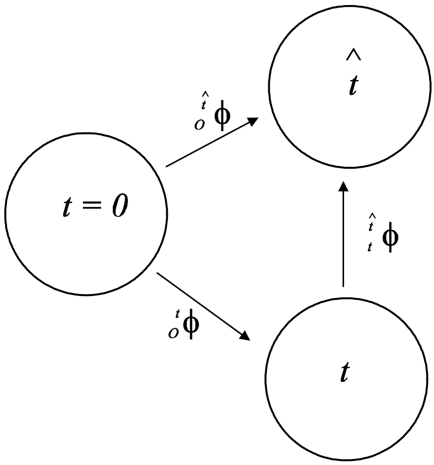

Fig. 2.7. Deformation processes between three configurations

$$ {}^{\hat t}_{\circ}\underline{\underline{\mathbf{X}}} \;=\; {}^{\hat t}_{t}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \,. \tag{2.130a}$$

For a *covariant Eulerian tensor* ${}^{t}\underline{\underline{\mathbf{t}}}$, that without losing generality we take as a second-order tensor:

$$ {}^{\hat t}t^{\hat\imath\hat\jmath} \;=\; {}^{\hat t}_{t}X^{\hat\imath}_{\;a}\, {}^{\hat t}_{t}X^{\hat\jmath}_{\;b}\, {}^{t}t^{ab} \,, \tag{2.130b}$$

$$ \left| {}^{\hat t}_{\circ}\phi^{*}\; {}^{\hat t}t^{\hat\imath\hat\jmath} \right|^{AB} \;=\; ({}^{\hat t}_{\circ}X^{-1})^{A}_{\;\hat\imath}\, ({}^{\hat t}_{\circ}X^{-1})^{B}_{\;\hat\jmath}\, {}^{\hat t}_{t}X^{\hat\imath}_{\;a}\, {}^{\hat t}_{t}X^{\hat\jmath}_{\;b}\, {}^{t}t^{ab} \,. \tag{2.130c}$$

Using Eq. (2.130a) in the above, we get

$$ \left| {}^{\hat t}_{\circ}\phi^{*}\; {}^{\hat t}t^{\hat\imath\hat\jmath} \right|^{AB} \;=\; ({}^{t}_{\circ}X^{-1})^{A}_{\;l}\, ({}^{t}_{\circ}X^{-1})^{B}_{\;k}\, {}^{t}t^{lk} \,, \tag{2.130d}$$

$$ \left| {}^{\hat t}_{\circ}\phi^{*}\; {}^{\hat t}t^{\hat\imath\hat\jmath} \right|^{AB} \;=\; \left| {}^{t}_{\circ}\phi^{*}\; {}^{t}t^{lk} \right|^{AB} \,. \tag{2.130e}$$

From the above and from Eq. (2.125) it follows that:

$$ \mathbf{L}_{{}^{t}\hat{\underline{\mathbf{v}}}}\left({}^{\hat t}\underline{\underline{\mathbf{t}}}\right)^{\hat a\hat b} \;=\; {}^{\hat t}_{t}X^{\hat a}_{\;l}\, {}^{\hat t}_{t}X^{\hat b}_{\;m} \left[\mathbf{L}_{{}^{t}\underline{\mathbf{v}}}\left({}^{t}\underline{\underline{\mathbf{t}}}\right)\right]^{lm} \;=\; \left\{ {}^{\hat t}_{t}\phi^{*} \left[\mathbf{L}_{{}^{t}\underline{\mathbf{v}}}\left({}^{t}\underline{\underline{\mathbf{t}}}\right)\right]^{lm} \right\}^{\hat a\hat b} \,. \tag{2.130f}$$

The above equality shows that the Lie derivative of a covariant Eulerian tensor is also a covariant Eulerian tensor.

Example 2.21. ◀◀◀◀◀
Considering again the case of a moving Cartesian frame and a fixed one from Example 2.15, we get

$$ {}^t v^\alpha \;=\; \dot{c}^\alpha \;+\; \dot{Q}^\alpha_{\;\gamma}\; ({}^t z^*)^\gamma \;+\; Q^\alpha_{\;\gamma}\; ({}^t v^*)^\gamma \;, $$

therefore taking into account that $\underline{\mathbf{c}}$, $\underline{\underline{\mathbf{Q}}}(t)$ and $\underline{\underline{\dot{\mathbf{Q}}}}(t)$, for the case under consideration, are constant in space,

$$ {}^t l^\alpha_{\;\beta} \;=\; \dot{Q}^\alpha_{\;\gamma}\; \frac{\partial\, ({}^t z^*)^\gamma}{\partial\, {}^t z^\beta} \;+\; Q^\alpha_{\;\gamma}\; ({}^t l^*)^\gamma_{\;\epsilon}\; \frac{\partial\, ({}^t z^*)^\epsilon}{\partial\, {}^t z^\beta} $$

hence,

$$ {}^t l^\alpha_{\;\beta} \;=\; \dot{Q}^\alpha_{\;\gamma}\; (Q^T)^\gamma_{\;\beta} \;+\; Q^\alpha_{\;\gamma}\; ({}^t l^*)^\gamma_{\;\epsilon}\; (Q^T)^\epsilon_{\;\beta} \;. $$

Comparing with Eq. (2.101c) it is obvious that ${}^t\underline{\underline{\mathbf{l}}}$ is not an objective tensor. Since the velocity gradient tensor is not objective in the classical sense we know that it is not a covariant tensor. ◀◀◀◀◀

Example 2.22. ◀◀◀◀◀
From Example 2.19, we know that ${}^t\underline{\underline{\mathbf{d}}}$ is the result of a Lie derivative; hence we can assess that the *Eulerian strain rate tensor is a covariant (objective) tensor.* ◀◀◀◀◀

Example 2.23. ◀◀◀◀◀
For a Eulerian tensor ${}^t\underline{\underline{\mathbf{t}}}$, we define in the reference configuration the tensor:

$$ {}^t\underline{\underline{\mathbf{T}}}^\sharp \;=\; \left({}^t_\circ X^{-1}\right)^A_{\;\;a}\; \left({}^t_\circ X^{-1}\right)^B_{\;\;b}\; {}^t t^{ab}\; {}^\circ\underline{\mathbf{g}}_A\; {}^\circ\underline{\mathbf{g}}_B $$

which can be written as

$$ {}^t\underline{\underline{\mathbf{T}}}^\sharp \;=\; {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \;\cdot\; {}^t\underline{\underline{\mathbf{t}}} \;\cdot\; {}^t_\circ\underline{\underline{\mathbf{X}}}^{-T} \qquad . $$

Since, ${}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \;\cdot\; {}^t_\circ\underline{\underline{\mathbf{X}}} \;=\; {}^\circ\underline{\underline{\mathbf{g}}}$ we can derive that,

$$ \frac{\mathrm{d}}{\mathrm{d}t}\; \left({}^t_\circ\underline{\underline{\mathbf{X}}}^{-1}\right) \;=\; -\; {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \;\cdot\; {}^t\underline{\underline{\mathbf{l}}} $$

and

$$ \frac{\mathrm{d}}{\mathrm{d}t}\; \left({}^t_\circ\underline{\underline{\mathbf{X}}}^{-T}\right) \;=\; -{}^t\,\underline{\underline{\mathbf{l}}}^T \;\cdot\; {}^t_\circ\underline{\underline{\mathbf{X}}}^{-T} $$

we can write

$$ \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\; \left({}^t\underline{\underline{\mathbf{T}}}^\sharp\right) \;=\;& {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \cdot {}^t\underline{\underline{\dot{\mathbf{t}}}} \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^{-T} \;-\; {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \cdot {}^t\underline{\underline{\mathbf{l}}} \cdot {}^t\underline{\underline{\mathbf{t}}} \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^{-T} \\ & -\; {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1} \cdot {}^t\underline{\underline{\mathbf{t}}} \cdot {}^t\underline{\underline{\mathbf{l}}}^T \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^{-T} \qquad . \end{aligned} $$

Also, from Eqs. (2.110c)

$$ {}^t\underline{\underline{\mathbf{l}}}^T = \underline{\boldsymbol{\nabla}}\, {}^t\underline{\mathbf{v}} = {}^tv^p|_n\, {}^t\underline{\mathbf{g}}^n\, {}^t\underline{\mathbf{g}}_p \quad . $$

Considering that the time derivative of the reference configuration base vectors is zero and using the above together with the Lie derivative definition in Eq. (2.125), we get

$$ \left[L_{{}^t\underline{\mathbf{v}}}\,({}^t\underline{\underline{\mathbf{t}}})\right]^{ab} = {}^t\dot{t}^{ab} - {}^tt^{nb}\, {}^tv^a|_n - {}^tt^{al}\, {}^tv^b|_l \ . $$

The above equation is going to be used in Sect. 3.4 for deriving objective stress rates. ◀◀◀◀◀

2.15 Compatibility

In our previous description of the kinematics of continuous media we went through the following path:

$$ \boxed{\text{Assume the existence of a regular mapping } {}^t\boldsymbol{\phi}} $$
$$ \Downarrow $$
$$ \boxed{\text{Calculate the tensorial components of different deformation measures}} $$

If, instead of the above, we want to start by defining the tensorial components of a given deformation measure, our freedom to define them is limited by the fact that they should guarantee the existence of a regular mapping from which they could be derived. The conditions that the tensorial components of a deformation measure should fulfill in order to assure the existence of a regular mapping are called their compatibility conditions.

In what follows we will derive the compatibility conditions for the Green deformation tensor.

From Eqs. (2.61), (2.80c) and (2.93a) we have,

Eulerian tensor	Spatial configuration	Pull-back space
Length differential	${}^t\mathrm{d}l$	${}^t\mathrm{d}l$
Coordinate differential	${}^t\mathrm{d}x^a$	$[({}^t\mathrm{d}x^a)^\sharp]^A = {}^\circ\mathrm{d}x^A$
Metric tensor	${}^tg_{ab}$	$[({}^tg_{ab})^\flat]_{AB} = {}^t_\circ C_{AB}$

Hence,

$$ {}^t\mathrm{d}x^a\, {}^tg_{ab}\, {}^t\mathrm{d}x^b = {}^\circ\mathrm{d}x^A\, {}^t_\circ C_{AB}\, {}^\circ\mathrm{d}x^B \ . \tag{2.131} $$

From the above equation it is obvious that the covariant components of the Green deformation tensor are the covariant components of the metric tensor

of the pull-back space. Note that the pull-back space is by no means coincident with the reference configuration, whose metric tensor has the covariant components ${}^{\circ}g_{AB}$.

Since we restrict our study of the kinematics of continuous media to the *Euclidean space*, we can assess that *the Riemann-Christoffel tensor is zero in the spatial configuration* (McConnell 1957). Hence,

$$ {}^{t}R_{prsq} \;=\; 0 \; . \tag{2.132} $$

The above equation represents 81 *compatibility conditions to be fulfilled in the spatial configuration.* However, the covariant components of the Riemann-Christoffel tensor satisfy the following relations (Aris 1962)[7]:

$$ {}^{t}R_{prsq} \;=\; -\;{}^{t}R_{rpsq} \; , \tag{2.133a} $$

$$ {}^{t}R_{prsq} \;=\; -\;{}^{t}R_{prqs} \; , \tag{2.133b} $$

$$ {}^{t}R_{prsq} \;=\; {}^{t}R_{sqpr} \; . \tag{2.133c} $$

We must also consider that ${}^{t}R_{iiii} = 0$; ${}^{t}R_{iiij} = 0$; ${}^{t}R_{iijj} = 0$ can be easily transformed into a trivial identity of the form $0 = 0$. Hence we are left with only 6 significant equations, namely:

$$ \begin{aligned} {}^{t}R_{1212} &= 0 \quad ; \quad {}^{t}R_{1213} = 0 \quad ; \quad {}^{t}R_{1223} = 0 \; ; \\ {}^{t}R_{1313} &= 0 \quad ; \quad {}^{t}R_{1323} = 0 \quad ; \quad {}^{t}R_{2323} = 0 \; . \end{aligned} \tag{2.134} $$

From Eqs. (A.79a-A.79e),

$$ \begin{aligned} {}^{t}R_{prsq} \;=\; & \frac{1}{2}\left(\frac{\partial^2\;{}^{t}g_{pq}}{\partial^{t}x^{r}\;\partial^{t}x^{s}} + \frac{\partial^2\;{}^{t}g_{rs}}{\partial^{t}x^{p}\;\partial^{t}x^{q}} - \frac{\partial^2\;{}^{t}g_{ps}}{\partial^{t}x^{r}\;\partial^{t}x^{q}} - \frac{\partial^2\;{}^{t}g_{rq}}{\partial^{t}x^{p}\;\partial^{t}x^{s}}\right) \\ & + {}^{t}g^{mn}\;\left({}^{t}\Gamma_{rsm}\;{}^{t}\Gamma_{pqn} \;-\; {}^{t}\Gamma_{rqm}\;{}^{t}\Gamma_{psn}\right) \end{aligned} \tag{2.135a} $$

where the ${}^{t}\Gamma_{ijk}$ are the Christoffel symbols of the first kind corresponding to the coordinate system $\{{}^{t}x^{a}\}$.

Hence, using Eq. (A.79b)

$$ \begin{aligned} {}^{t}R_{prsq} \;=\; & \frac{1}{2}\left(\frac{\partial^2\;{}^{t}g_{pq}}{\partial^{t}x^{r}\;\partial^{t}x^{s}} + \frac{\partial^2\;{}^{t}g_{rs}}{\partial^{t}x^{p}\;\partial^{t}x^{q}} - \frac{\partial^2\;{}^{t}g_{ps}}{\partial^{t}x^{r}\;\partial^{t}x^{q}} - \frac{\partial^2\;{}^{t}g_{rq}}{\partial^{t}x^{p}\;\partial^{t}x^{s}}\right) \\ & + {}^{t}g^{mn}\left[\frac{1}{4}\left(\frac{\partial\;{}^{t}g_{sm}}{\partial^{t}x^{r}} + \frac{\partial\;{}^{t}g_{mr}}{\partial^{t}x^{s}} - \frac{\partial\;{}^{t}g_{rs}}{\partial^{t}x^{m}}\right)\left(\frac{\partial\;{}^{t}g_{qn}}{\partial^{t}x^{p}} + \frac{\partial\;{}^{t}g_{np}}{\partial^{t}x^{q}} - \frac{\partial\;{}^{t}g_{pq}}{\partial^{t}x^{n}}\right)\right. \\ & \left. -\frac{1}{4}\left(\frac{\partial\;{}^{t}g_{qm}}{\partial^{t}x^{r}} + \frac{\partial\;{}^{t}g_{mr}}{\partial^{t}x^{q}} - \frac{\partial\;{}^{t}g_{rq}}{\partial^{t}x^{m}}\right)\left(\frac{\partial\;{}^{t}g_{sn}}{\partial^{t}x^{p}} + \frac{\partial\;{}^{t}g_{np}}{\partial^{t}x^{s}} - \frac{\partial\;{}^{t}g_{ps}}{\partial^{t}x^{n}}\right)\right] = 0 \; . \end{aligned} \tag{2.135b} $$

[7] See Appendix.

Doing a pull-back operation on Eq. (2.132) we obtain,

$$\left[{}^t\phi^*\left({}^tR_{prsq}\right)\right]_{PRSQ} = {}^t_\circ X^p{}_P\,{}^t_\circ X^r{}_R\,{}^t_\circ X^s{}_S\,{}^t_\circ X^q{}_Q\,{}^tR_{prsq} = 0\,. \tag{2.136a}$$

Example 2.24. ◀◀◀◀◀

Equation (2.135b) represents the components of the tensorial equation

$$ {}^t\underline{\underline{\underline{\underline{\mathbf{R}}}}} = \underline{\underline{\underline{\underline{\mathbf{0}}}}} \quad . $$

If in the spatial configuration we change from the $\{{}^tx^i\}$ coordinate system to the $\{{}^t\tilde{x}^i\}$ system, we write Eq. (2.135b) using,

$$ {}^t\tilde{g}_{pq} = {}^tg_{lm}\,\frac{\partial^tx^l}{\partial^t\tilde{x}^p}\,\frac{\partial^tx^m}{\partial^t\tilde{x}^q} $$

$$ {}^t\tilde{g}^{pq} = {}^tg^{lm}\,\frac{\partial^t\tilde{x}^p}{\partial^tx^l}\,\frac{\partial^t\tilde{x}^q}{\partial^tx^m} $$

and the equation would look like

$$ {}^t\tilde{R}_{prsq} = \frac{1}{2}\left(\frac{\partial^2\,{}^t\tilde{g}_{pq}}{\partial^t\tilde{x}^r\partial^t\tilde{x}^s} + \cdots\right) + {}^t\tilde{g}^{mn}\left[\frac{1}{4}\left(\frac{\partial^t\tilde{g}_{sm}}{\partial^t\tilde{x}^r} + \cdots\right)\right] = 0 \quad . $$

If we now want to do a pull-back of Eq.(2.135b) the algebra can get quite lengthy, but we can use an analogy with the above tensor transformations:

$$ \left[{}^t\phi^*({}^tg_{lm})_{PQ}\right] = {}^tg_{lm}\,\frac{\partial^tx^l}{\partial^\circ x^P}\,\frac{\partial^tx^m}{\partial^\circ x^Q} = {}^t_\circ C_{PQ} $$

$$ \left[{}^t\phi^*({}^tg^{lm})\right]^{PQ} = {}^tg^{lm}\,\frac{\partial^\circ x^P}{\partial^tx^l}\,\frac{\partial^\circ x^Q}{\partial^tx^m} = \left({}^t_\circ C^{-1}\right)^{PQ} \quad . $$

Using this formal analogy we can easily write:

$$ \begin{aligned} \left[{}^t\phi^*\left({}^tR_{prsq}\right)\right]_{PRSQ} = {}& \frac{1}{2}\left[\frac{\partial^2\,{}^t_\circ C_{PQ}}{\partial^\circ x^R\partial^\circ x^S} + \frac{\partial^2\,{}^t_\circ C_{RS}}{\partial^\circ x^P\partial^\circ x^Q} - \frac{\partial^2\,{}^t_\circ C_{PS}}{\partial^\circ x^R\partial^\circ x^Q}\right. \\ & \left. - \frac{\partial^2\,{}^t_\circ C_{RQ}}{\partial^\circ x^P\partial^\circ x^S}\right] + \left({}^t_\circ C^{-1}\right)^{MN} \\ & \left[\frac{1}{4}\left(\frac{\partial\,{}^t_\circ C_{SM}}{\partial^\circ x^R} + \frac{\partial\,{}^t_\circ C_{MR}}{\partial^\circ x^S} - \frac{\partial\,{}^t_\circ C_{RS}}{\partial^\circ x^M}\right)\left(\frac{\partial\,{}^t_\circ C_{QN}}{\partial^\circ x^P} + \frac{\partial\,{}^t_\circ C_{NP}}{\partial^\circ x^Q} - \frac{\partial\,{}^t_\circ C_{PQ}}{\partial^\circ x^N}\right)\right. \\ & \left. -\frac{1}{4}\left(\frac{\partial\,{}^t_\circ C_{QM}}{\partial^\circ x^R} + \frac{\partial\,{}^t_\circ C_{MR}}{\partial^\circ x^Q} - \frac{\partial\,{}^t_\circ C_{RQ}}{\partial^\circ x^M}\right)\left(\frac{\partial\,{}^t_\circ C_{SN}}{\partial^\circ x^P} + \frac{\partial\,{}^t_\circ C_{NP}}{\partial^\circ x^S} - \frac{\partial\,{}^t_\circ C_{PS}}{\partial^\circ x^N}\right)\right] \end{aligned} $$

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Hence,

$$\frac{1}{2}\left[\frac{\partial^2\,{}^t_\circ C_{PQ}}{\partial^\circ x^R \partial^\circ x^S} + \frac{\partial^2\,{}^t_\circ C_{RS}}{\partial^\circ x^P \partial^\circ x^Q} - \frac{\partial^2\,{}^t_\circ C_{PS}}{\partial^\circ x^R \partial^\circ x^Q} - \frac{\partial^2\,{}^t_\circ C_{RQ}}{\partial^\circ x^P \partial^\circ x^S}\right] + \left({}^t_\circ C^{-1}\right)^{MN} \tag{2.136b}$$

$$\left[\frac{1}{4}\left(\frac{\partial\,{}^t_\circ C_{SM}}{\partial^\circ x^R} + \frac{\partial\,{}^t_\circ C_{MR}}{\partial^\circ x^S} - \frac{\partial\,{}^t_\circ C_{RS}}{\partial^\circ x^M}\right)\left(\frac{\partial\,{}^t_\circ C_{QN}}{\partial^\circ x^P} + \frac{\partial\,{}^t_\circ C_{NP}}{\partial^\circ x^Q} - \frac{\partial\,{}^t_\circ C_{PQ}}{\partial^\circ x^N}\right)\right.$$

$$\left. -\frac{1}{4}\left(\frac{\partial\,{}^t_\circ C_{QM}}{\partial^\circ x^R} + \frac{\partial\,{}^t_\circ C_{MR}}{\partial^\circ x^Q} - \frac{\partial\,{}^t_\circ C_{RQ}}{\partial^\circ x^M}\right)\left(\frac{\partial\,{}^t_\circ C_{SN}}{\partial^\circ x^P} + \frac{\partial\,{}^t_\circ C_{NP}}{\partial^\circ x^S} - \frac{\partial\,{}^t_\circ C_{PS}}{\partial^\circ x^N}\right)\right] = 0.$$

We can now define in the pull-back space, with metric ${}^t_\circ C_{AB}$, the Christoffel symbols of the first kind:

$$^*\Gamma_{RSM} = \frac{1}{2}\left(\frac{\partial^t_\circ C_{SM}}{\partial^\circ x^R} + \frac{\partial^t_\circ C_{MR}}{\partial^\circ x^S} - \frac{\partial^t_\circ C_{RS}}{\partial^\circ x^M}\right) \tag{2.136c}$$

therefore we obtain the following 6 *compatibility conditions* for the covariant components of the Green deformation tensor:

$$\phi^*({}^tR_{prsq}) = \frac{1}{2}\left(\frac{\partial^2\,{}^t_\circ C_{PQ}}{\partial^\circ x^R \partial^\circ x^S} + \frac{\partial^2\,{}^t_\circ C_{RS}}{\partial^\circ x^P \partial^\circ x^Q} - \frac{\partial^2\,{}^t_\circ C_{PS}}{\partial^\circ x^R \partial^\circ x^Q} - \frac{\partial^2\,{}^t_\circ C_{RQ}}{\partial^\circ x^P \partial^\circ x^S}\right)$$

$$+ \left({}^t_\circ C^{-1}\right)^{MN}\left({}^*\Gamma_{RSM}\,{}^*\Gamma_{PQN} - {}^*\Gamma_{RQM}\,{}^*\Gamma_{PSN}\right) = 0\,. \tag{2.136d}$$

The above equation indicates that ${}^t_\circ C_{AB}$ is a metric of the Euclidean space.

Taking into account the Bianchi identities in Eq. (A.82d) (Synge & Schild 1949) and Eqs.(2.135a), we get

$$\begin{aligned} {}^tR_{1212|3} + {}^tR_{1213|1} + {}^tR_{1213|2} &= 0\,, \\ {}^tR_{1313|2} + {}^tR_{1323|1} - {}^tR_{1213|3} &= 0\,, \\ {}^tR_{2323|1} - {}^tR_{1323|2} + {}^tR_{1223|3} &= 0\,, \end{aligned} \tag{2.137}$$

and we reduce the number of *independent compatibility conditions* to 3.

Example 2.25. ◀◀◀◀◀

Using as a metric the Green tensor, we can define an analog to Eq.(A.59):

$$\mathbf{GRAD}_{{}^t_\circ\underline{\underline{\mathbf{C}}}}\left({}^t\underline{\mathbf{A}}^\flat\right) = \left[\frac{\partial^t A^\flat_I}{\partial^\circ x^A} - {}^tA^\flat_D\,{}^*\Gamma^D_{IA}\right]{}^\circ\underline{\mathbf{g}}^I\,{}^\circ\underline{\mathbf{g}}^A$$

where for the Eulerian vector ${}^t\underline{\mathbf{a}}$,

$$\left({}^tA^\flat\right)_I = {}^t_\circ X^i{}_I\; a_i\,,$$

and,

$$ {}^{*}\Gamma^{D}_{IA} \;=\; \frac{1}{2}\;\left({}^{t}_{\circ}C^{-1}\right)^{DK}\;\left[\frac{\partial^{t}_{\circ}C_{IK}}{\partial^{\circ}x^{A}} \;+\; \frac{\partial^{t}_{\circ}C_{AK}}{\partial^{\circ}x^{I}} \;-\; \frac{\partial^{t}_{\circ}C_{IA}}{\partial^{\circ}x^{K}}\right]\;. $$

Taking into account that

$$ \begin{aligned} {}^{t}_{\circ}C_{IK} \;&=\; \frac{\partial^{t}x^{r}}{\partial^{\circ}x^{I}}\;\frac{\partial^{t}x^{s}}{\partial^{\circ}x^{K}}\;{}^{t}g_{rs}\;, \\ \left({}^{t}_{\circ}C^{-1}\right)^{DK} \;&=\; \frac{\partial^{\circ}x^{D}}{\partial^{t}x^{r}}\;\frac{\partial^{\circ}x^{K}}{\partial^{t}x^{s}}\;{}^{t}g^{rs}\;, \end{aligned} $$

it is easy to show that:

$$ \left[\phi^{*}\;\left(\underline{\boldsymbol{\nabla}}\;{}^{t}\underline{\mathbf{a}}\right)_{ia}\right]_{IA} \;=\; \left[\mathbf{GRAD}_{{}^{t}_{\circ}\underline{\underline{\mathbf{C}}}}\;\left({}^{t}\underline{\mathbf{A}}^{\flat}\right)\right]_{IA}\;. $$

Example 2.26.

In a Cartesian coordinate system, from Eqs. (2.9a-2.9b), (2.29a-2.29c), (2.65) and (2.68) we get

$$ \begin{aligned} {}^{t}_{\circ}\varepsilon_{\alpha\beta} \;&=\; \frac{1}{2}\;\left(\frac{\partial^{t}u^{\alpha}}{\partial^{\circ}z^{\beta}} \;+\; \frac{\partial^{t}u^{\beta}}{\partial^{\circ}z^{\alpha}} \;+\; \frac{\partial^{t}u^{\gamma}}{\partial^{\circ}z^{\alpha}}\;\frac{\partial^{t}u^{\gamma}}{\partial^{\circ}z^{\beta}}\right)\;, \\ {}^{t}e_{\alpha\beta} \;&=\; \frac{1}{2}\;\left(\frac{\partial^{t}u^{\alpha}}{\partial^{t}z^{\beta}} \;+\; \frac{\partial^{t}u^{\beta}}{\partial^{t}z^{\alpha}} \;-\; \frac{\partial^{t}u^{\gamma}}{\partial^{t}z^{\alpha}}\;\frac{\partial^{t}u^{\gamma}}{\partial^{t}z^{\beta}}\right)\;. \end{aligned} $$

For linear kinematics,

$$ \frac{\partial^{t}u^{\alpha}}{\partial^{\circ}z^{\beta}} \;<<\; 1\;, $$

and also

$$ \frac{\partial(.)}{\partial^{\circ}z^{\beta}} \;\cong\; \frac{\partial(.)}{\partial^{t}z^{\beta}} \qquad . $$

Therefore, for linear kinematics

$$ {}^{t}_{\circ}\varepsilon_{\alpha\beta} \;\cong \frac{1}{2}\;\left(\frac{\partial^{t}u^{\alpha}}{\partial^{\circ}z^{\beta}} \;+\; \frac{\partial^{t}u^{\beta}}{\partial^{\circ}z^{\alpha}}\right) \;=\; {}^{t}\varepsilon_{\alpha\beta}\;, $$

where we call ${}^{t}\varepsilon_{\alpha\beta}$ the components of the infinitesimal strain tensor.
From Eq. (2.65) we know that:

$$ {}^{t}_{\circ}\varepsilon_{\alpha\beta} \;=\; \frac{1}{2}\;\left({}^{t}_{\circ}C_{\alpha\beta} \;-\; \delta_{\alpha}\beta\right)\;. $$

Introducing the above in Eq. (2.135b) and linearizing (neglecting the higher powers of ${}^{t}_{\circ}\varepsilon_{\alpha\beta}$) we get the compatibility equations corresponding to the assumption of linear kinematics:

$$\frac{\partial^2 \ {}^t\varepsilon_{\alpha\beta}}{\partial^\circ z^\gamma \ \partial^\circ z^\delta} + \frac{\partial^2 \ {}^t\varepsilon_{\gamma\delta}}{\partial^\circ z^\alpha \ \partial^\circ z^\beta} - \frac{\partial^2 \ {}^t\varepsilon_{\alpha\delta}}{\partial^\circ z^\gamma \ \partial^\circ z^\beta} - \frac{\partial^2 \ {}^t\varepsilon_{\gamma\beta}}{\partial^\circ z^\alpha \ \partial^\circ z^\delta} = 0 \ .$$

The above represents a set of 6 equations, that proceeding as in Eqs. (2.137), can be reduced to 3 independent compatibility conditions.
The above result was obtained for a Cartesian system. Generalizing for an arbitrary coordinate system we get

$${}^t\varepsilon_{AB|_{CD}} + {}^t\varepsilon_{CD|_{AB}} - {}^t\varepsilon_{AD|_{BC}} - {}^t\varepsilon_{BC|_{AD}} = 0 \ .$$

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3

Stress Tensor

To deform a continuous body the exterior medium has to produce a loading on that body; therefore we get *external forces* acting on it. Also, during the deformation of a continuous body, neighboring particles exert forces on each other, they are the *internal forces* in the body.

The study of the internal forces in a body leads to the notion of *stresses*, that we are going to develop in this chapter.

Some reference books for this chapter are: (Truesdell & Noll 1965, Truesdell 1966, Malvern 1969, Marsden & Hughes 1983).

3.1 External forces

When studying the deformation of a continuous body, to classify a force as either external or internal, we have to carefully take into account our definition of the body under consideration.

For example, in Fig. 3.1 *a*, we study the spatial configuration at t of the body $({}^{t}\Omega_A \cup {}^{t}\Omega_B)$ with the external forces ${}^{t}\underline{\mathbf{F}}_A$ and ${}^{t}\underline{\mathbf{F}}_B$ and in Fig. 3.1 *b*, to study the same physical problem, we choose to consider ${}^{t}\Omega_A$ and ${}^{t}\Omega_B$ as *free bodies*. Obviously both analyses will lead to the same conclusions but, it is clear that in the first case ${}^{t}\underline{\mathbf{F}}_{AB}$ and ${}^{t}\underline{\mathbf{F}}_{BA}$ have to be considered as internal forces and in the second case as external forces.

We can find different types of *external forces*, for example:

- External forces acting on the elements of volume or mass inside the body (they are defined per unit volume or per unit mass). Theses forces are called *body forces* and some examples are: gravitational forces, inertia forces, electromagnetic forces, etc. (in general they are "action-at-a-distance" forces (Malvern 1969).

  Let us define in the t-configuration of a body a Cartesian system $\{{}^{t}\hat{z}^{\alpha}\ ,\ \alpha = 1,2,3\}$ with base vectors ${}^{t}\underline{\hat{\mathbf{e}}}_{\alpha}$ and an arbitrary curvilinear system $\{{}^{t}x^{i}\ ,\ i = 1,2,3\}$ with covariant base vectors ${}^{t}\underline{\mathbf{g}}_{i}$. We consider the vector

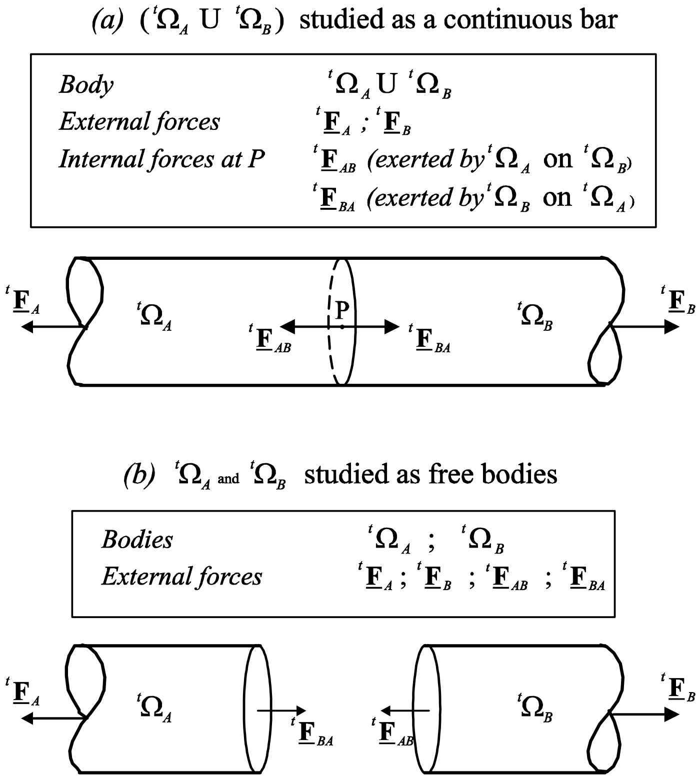


Fig. 3.1. External and internal forces in a bar

field ${}^t\underline{\mathbf{b}}({}^t\underline{\mathbf{x}})$ to define the forces per unit mass acting on the body at t . We can write

$$ {}^t\underline{\mathbf{b}} \;=\; {}^t\hat{b}^\alpha \; {}^t\underline{\hat{\mathbf{e}}}_\alpha \;=\; {}^tb^i \; {}^t\underline{\mathbf{g}}_i \; . \tag{3.1} $$

The resultant of the forces per unit mass is:

$$ \int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{b}} \; {}^t\mathrm{d}V \;=\; \int_{{}^tV} {}^t\rho \; {}^t\hat{b}^\alpha \; {}^t\underline{\hat{\mathbf{e}}}_\alpha \; {}^t\mathrm{d}V \;=\; \int_{{}^tV} {}^t\rho \; {}^tb^i \; {}^t\underline{\mathbf{g}}_i \; {}^t\mathrm{d}V \tag{3.2} $$

where, ${}^t\rho$: density of the body in t-configuration and ${}^tV$: volume of the body in the t-configuration.

- External forces acting on the elements of the body's surface (they are defined per unit surface). These forces are called *surface forces* and some examples are: pressure forces, contact forces, friction forces, etc. We consider the vector field ${}^t\underline{\mathbf{t}}({}^t\underline{\mathbf{x}})$ to define the forces per unit surface acting on the region ${}^tS_\sigma$, a subset of the surface of the body at t . We can write

$$ {}^t\underline{\mathbf{t}} \;=\; {}^t\hat{t}^{\alpha}\; {}^t\underline{\hat{\mathbf{e}}}_{\alpha} \;=\; {}^t t^i\; {}^t\underline{\mathbf{g}}_i \;. \tag{3.3} $$

The resultant of the forces per unit surface is:

$$ \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}}\; {}^t\mathrm{d}S \;=\; \int_{{}^tS_\sigma} {}^t\hat{t}^{\alpha}\; {}^t\underline{\hat{\mathbf{e}}}_{\alpha}\; {}^t\mathrm{d}S \;=\; \int_{{}^tS_\sigma} {}^t t^i\; {}^t\underline{\mathbf{g}}_i\; {}^t\mathrm{d}S \;. \tag{3.4} $$

It is important to point out that our description of the external forces acting on a body excludes the possibility of considering distributed torques per unit volume, mass or surface. Therefore, the moment with respect to a point P of the considered external forces in the t-configuration is:

$$ {}^t\underline{\mathbf{M}}_P \;=\; \int_{{}^tV} {}^t\rho\; \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}} \right)\; {}^t\mathrm{d}V \;+\; \int_{{}^tS_\sigma} \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{t}} \right)\; {}^t\mathrm{d}S \tag{3.5} $$

where ${}^t\underline{\mathbf{r}}$ is the vector that in the t-configuration goes from the moment center P to a point inside the body (first term on the r.h.s.) or to a point on the body surface (second term on the r.h.s.).

3.2 The Cauchy stress tensor

In Fig. 3.2 a we represent the spatial configuration of a body $\mathcal{B}$ corresponding to a time t, and we identify a particle P.

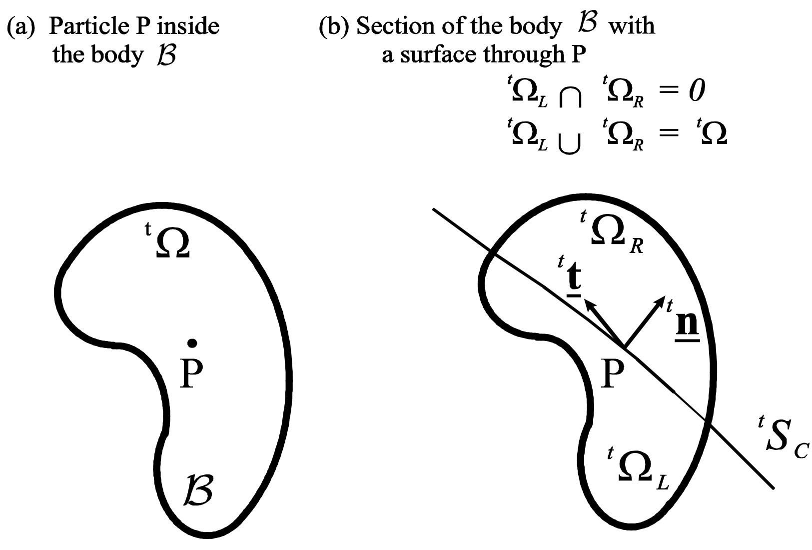


Fig. 3.2. Internal forces at a point inside a continuum

The external forces acting on $\mathcal{B}$ per unit mass are given by the vector field ${}^t\underline{\mathbf{b}}$ and the external forces per unit surface are given by the vector field ${}^t\underline{\mathbf{t}}$.

We now *section* the body $\mathcal{B}$, in the t-configuration, with a surface ${}^t S_c$ passing through P. The normal to the surface ${}^t S_c$ at P is ${}^t\underline{\mathbf{n}}$ (see Fig. 3.2 *b*).

If we now analyze in Fig. 3.2 *b* the left part, ${}^t\Omega_L$, as a free body, we have to consider as external forces the internal forces at P in Fig. 3.2 *a*.

Considering on the surface ${}^t S_c$ an area ${}^t\Delta S$ around P, the set of external forces acting on ${}^t\Delta S$ can be reduced to a force ${}^t\Delta\underline{\mathbf{F}}$ through P and a moment ${}^t\Delta\underline{\mathbf{M}}_P$.

When ${}^t\Delta S \to 0$:

$$\lim_{{}^t\Delta S \to 0} \frac{{}^t\Delta\underline{\mathbf{F}}}{{}^t\Delta S} = {}^t\underline{\mathbf{t}} \tag{3.6a}$$

$$\lim_{{}^t\Delta S \to 0} \frac{{}^t\Delta\underline{\mathbf{M}}_P}{{}^t\Delta S} = \underline{\mathbf{0}} \tag{3.6b}$$

The vector ${}^t\underline{\mathbf{t}}$ is known in the literature as *traction*.

Equations(3.6a-3.6b) incorporate two fundamental hypotheses:

- The limit in Eq. (3.6a) *exists*. Therefore we exclude from the continuum mechanics field the consideration of concentrated forces (concentrated forces are also not physically possible).
- The condition in Eq. (3.6b) is a strong requirement in the classical formulation of continuum mechanics. There are alternative formulations that do not require the fulfillment of Eq. (3.6b) (e.g. the theory of polar media (Truesdell & Noll 1965, Malvern 1969).

In this book we limit our study to the classical case of *non-polar media*.

It is interesting to note that from Eq. (3.6a) we can assess that, if we consider different surfaces through P, which share the external normal ${}^t\underline{\mathbf{n}}$ (tangent surfaces), we will arrive at the same traction vector ${}^t\underline{\mathbf{t}}$ (see Fig. 3.3).

We can define, in the t-configuration at the point P, a second-order tensor ${}^t\underline{\underline{\boldsymbol{\sigma}}}$, *the Cauchy stress tensor*, via the following equation:

$${}^t\underline{\mathbf{t}} = {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} . \tag{3.7}$$

Since ${}^t\underline{\mathbf{t}}$ and ${}^t\underline{\mathbf{n}}$ are vectors, using the quotient rule (Sect. A.5), it is evident that ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ is a second-order tensor.

We can consider Eq. (3.7) to be a condition of equivalence between external forces and stresses inside a continuum.

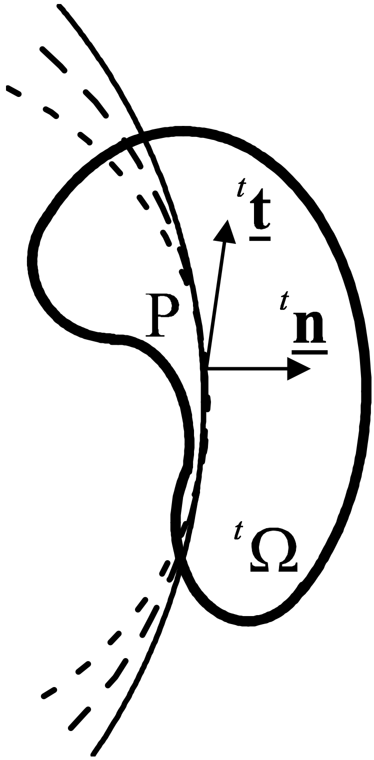


Fig. 3.3. Tangent surfaces at P have the same traction vector

3.2.1 Symmetry of the Cauchy stress tensor (Cauchy Theorem)

In Sect. 4.4.2 we will prove that *from the equilibrium equations of a nonpolar continuum* we get

$$^t\underline{\underline{\boldsymbol{\sigma}}} = {}^t\underline{\underline{\boldsymbol{\sigma}}}^T , \tag{3.8}$$

that is to say, the *Cauchy stress tensor is symmetric.*

Example 3.1. ____◀◀◀◀◀

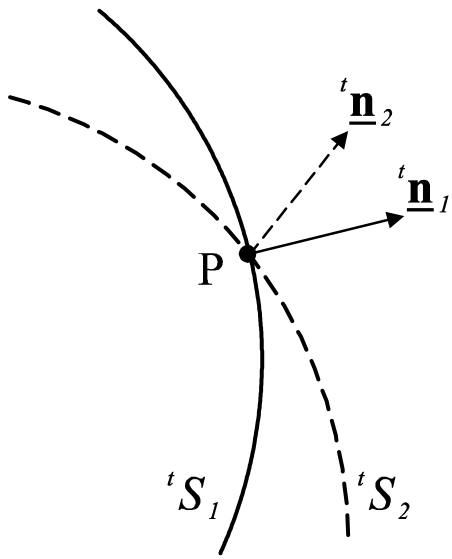
In the following figure, at the point P, inside the t-configuration of a continuum body, the stress tensor is $^t\underline{\underline{\boldsymbol{\sigma}}}$.

Cutting the continuum with the surface $^tS_1$, we get at P the traction vector

$$^t\underline{\mathbf{t}}_1 = {}^t\underline{\mathbf{n}}_1 \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} ,$$

if we cut with $^tS_2$, the traction vector at P is:

$$^t\underline{\mathbf{t}}_2 = {}^t\underline{\mathbf{n}}_2 \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} .$$



Secant surfaces at a point inside a continuum

In an arbitrary coordinate system $\{^t x_i\}$ we can write

$$\begin{aligned} {^t\underline{\mathbf{t}}_1} \cdot {^t\underline{\mathbf{n}}_2} &= {^t n_{1_i}}\, {^t\sigma^{ij}}\, {^t n_{2_j}} \;, \\ {^t\underline{\mathbf{t}}_2} \cdot {^t\underline{\mathbf{n}}_1} &= {^t n_{2_i}}\, {^t\sigma^{ij}}\, {^t n_{1_j}} \;, \end{aligned}$$

and since from Eq. (3.8) ${^t\sigma^{ij}} = {^t\sigma^{ji}}$ we get

$$ {^t\underline{\mathbf{t}}_1} \cdot {^t\underline{\mathbf{n}}_2} = {^t\underline{\mathbf{t}}_2} \cdot {^t\underline{\mathbf{n}}_1} \;. $$

The above result, a direct consequence of the Cauchy Theorem, is known as the *projection theorem* or *reciprocal theorem of Cauchy* (Malvern 1969).◀◀◀◀◀

3.3 Conjugate stress/strain rate measures

Let us assume, at an instant (load level) t, a continuum body $\mathcal{B}$ in equilibrium under the action of external body forces ${^t\underline{\mathbf{b}}}$ and external surface forces ${^t\underline{\mathbf{t}}}$.

Assuming a velocity field ${^t\underline{\mathbf{v}}}({^t\underline{\mathbf{x}}})$ on $\mathcal{B}$, the *power provided by the external forces* is:

$$ {^tP_{ext}} = \int_{^tV} {^t\rho}\, {^t\underline{\mathbf{b}}} \cdot {^t\underline{\mathbf{v}}}\, {^t\mathrm{d}V} + \int_{^tS} {^t\underline{\mathbf{t}}} \cdot {^t\underline{\mathbf{v}}}\, {^t\mathrm{d}S} \;. \qquad (3.9a) $$

Using Eq. (3.7) we can rewrite Eq.(3.9a) as

$$ {}^{t}P_{ext} = \int_{{}^{t}V} {}^{t}\rho\, {}^{t}\underline{\mathbf{b}} \cdot {}^{t}\underline{\mathbf{v}}\, {}^{t}\mathrm{d}V + \int_{{}^{t}S} \left({}^{t}\underline{\mathbf{n}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right) \cdot {}^{t}\underline{\mathbf{v}}\, {}^{t}\mathrm{d}S \,. \tag{3.9b} $$

From the Divergence Theorem (Hildebrand 1976),

$$ {}^{t}P_{ext} = \int_{{}^{t}V} \left[{}^{t}\rho\, {}^{t}\underline{\mathbf{b}} \cdot {}^{t}\underline{\mathbf{v}} + \underline{\boldsymbol{\nabla}} \cdot \left({}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t}\underline{\mathbf{v}} \right) \right] {}^{t}\mathrm{d}V \tag{3.9c} $$

introducing Eq. (2.110a-2.110c) and after some algebra,

$$ {}^{t}P_{ext} = \int_{{}^{t}V} \left[{}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{l}}} + \left({}^{t}\rho\, {}^{t}\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right) \cdot {}^{t}\underline{\mathbf{v}} \right] {}^{t}\mathrm{d}V \,. \tag{3.9d} $$

Since the t-configuration is an equilibrium configuration, the following equation (to be proved in Chap. 4, Eq.(4.27b)) holds

$$ {}^{t}\rho\, {}^{t}\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t}\rho\, \frac{D^{t}\underline{\mathbf{v}}}{Dt} \,, \tag{3.9e} $$

and an obvious result that we also need is:

$$ \frac{D^{t}\underline{\mathbf{v}}}{Dt} \cdot {}^{t}\underline{\mathbf{v}} = \frac{D}{Dt}\left[\frac{1}{2}\, {}^{t}\underline{\mathbf{v}} \cdot {}^{t}\underline{\mathbf{v}}\right] . \tag{3.9f} $$

The *kinetic energy* of the body $\mathcal{B}$, at the instant t, is defined as

$$ {}^{t}K = \int_{{}^{t}V} \frac{{}^{t}\rho}{2}\, {}^{t}\underline{\mathbf{v}} \cdot {}^{t}\underline{\mathbf{v}}\, {}^{t}\mathrm{d}V = \int_{m} \frac{1}{2}\, {}^{t}\underline{\mathbf{v}} \cdot {}^{t}\underline{\mathbf{v}}\, \mathrm{d}m \,. \tag{3.9g} $$

In the second integral of the above equation we integrate over the mass of the body $\mathcal{B}$. Since the mass of the body is invariant,

$$ \frac{D^{t}K}{Dt} = \int_{m} \frac{D^{t}\underline{\mathbf{v}}}{Dt} \cdot {}^{t}\underline{\mathbf{v}}\, \mathrm{d}m \,. \tag{3.9h} $$

Finally, using the decomposition of the velocity gradient tensor in Eq. (2.112a) and considering that since $({}^{t}\underline{\underline{\boldsymbol{\sigma}}})$ is a symmetric tensor and $({}^{t}\underline{\underline{\boldsymbol{\omega}}})$ is a skew-symmetric one,

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\boldsymbol{\omega}}} = 0 \tag{3.9i} $$

we get

$$ {}^{t}P_{ext} = \frac{D^{t}K}{Dt} + \int_{{}^{t}V} {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{d}}}\, {}^{t}\mathrm{d}V \,. \tag{3.9j} $$

We define

$$ {}^{t}P_{\sigma} = \int_{{}^{t}V} {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{d}}}\, {}^{t}\mathrm{d}V \tag{3.10} $$

as the *stresses power*. Obviously ${}^{t}P_{\sigma}$ is the fraction of ${}^{t}P_{ext}$ that is not transformed into kinetic energy and that is either stored in the body material or dissipated by the body material, depending on its properties (see Chapter 5).

From Eq. (3.10) we define the spatial tensors ${}^{t}\underline{\underline{\boldsymbol{\sigma}}}$ and ${}^{t}\underline{\underline{\mathbf{d}}}$ to be *energy conjugate* (Atluri 1984).

In what follows we will define other pairs of energy conjugate stress/strain rate measures.

3.3.1 The Kirchhoff stress tensor

From Eqs. (3.10) and (2.34d) and the *mass-conservation principle* (to be discussed in Chapter 4, Eq.(4.20d)) we get

$$ {}^tP_\sigma = \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}}\ {}^t\mathrm{d}V = \int_{{}^\circ V} \frac{{}^\circ\rho}{{}^t\rho}\ {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}}\ {}^\circ\mathrm{d}V \ . \tag{3.11} $$

The *Kirchhoff stress tensor* is defined as

$$ {}^t\underline{\underline{\boldsymbol{\tau}}} = \frac{{}^\circ\rho}{{}^t\rho}\ {}^t\underline{\underline{\boldsymbol{\sigma}}} \tag{3.12} $$

where, ${}^\circ\rho$: density in the reference configuration and ${}^\circ V$: volume of the reference configuration.

It is important to note that although the Kirchhoff stress tensor was introduced by calculating ${}^tP_\sigma$ via an integral defined over the reference volume, Eq. (3.12) clearly shows that ${}^t\underline{\underline{\boldsymbol{\tau}}}$ is defined in the same space where ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ is defined: the *spatial configuration*. Hence, using in the t-configuration an arbitrary curvilinear coordinate system $\{{}^tx^a\}$ with covariant base vectors ${}^t\underline{\mathbf{g}}_a$ we obtain

$$ {}^t\underline{\underline{\boldsymbol{\tau}}} = {}^t\tau^{ab}\ {}^t\underline{\mathbf{g}}_a\ {}^t\underline{\mathbf{g}}_b \tag{3.13a} $$

$$ {}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^t\sigma^{ab}\ {}^t\underline{\mathbf{g}}_a\ {}^t\underline{\mathbf{g}}_b \tag{3.13b} $$

and

$$ {}^t\tau^{ab} = \frac{{}^\circ\rho}{{}^t\rho}\ {}^t\sigma^{ab} \ . \tag{3.13c} $$

3.3.2 The first Piola-Kirchhoff stress tensor

From Eqs. (3.11) and (3.9i) we can write

$$ {}^tP_\sigma = \int_{{}^\circ V} {}^t\underline{\underline{\boldsymbol{\tau}}} : {}^t\underline{\underline{\mathbf{l}}}\ {}^\circ\mathrm{d}V \ . \tag{3.14} $$

In Chap. 2 we learned how to derive representations in the reference configuration of tensors defined in the spatial configuration via pull-back operations. We will now obtain a representation of the Kirchhoff stress tensor in the form of a *two-point tensor*.

In the reference configuration we define an arbitrary coordinate system $\{{}^\circ x^A\}$ with covariant base vectors ${}^\circ\underline{\mathbf{g}}_A$; and in the spatial configuration a system $\{{}^tx^a\}$ with covariant base vectors ${}^t\underline{\mathbf{g}}_a$. We also define a convected system $\{\theta^i\}$ with covariant base vectors in the reference configuration ${}^\circ\underline{\widetilde{\mathbf{g}}}_i$ and covariant base vectors in the spatial configuration ${}^t\underline{\widetilde{\mathbf{g}}}_i$.

In the spatial configuration we can write the Kirchhoff stress tensor as

$$ {}^{t}\underline{\underline{\boldsymbol{\tau}}} = {}^{t}\tilde{\tau}^{ij}\ {}^{t}\underline{\tilde{\mathbf{g}}}_i\ {}^{t}\underline{\tilde{\mathbf{g}}}_j \tag{3.15a} $$

a pull-back of the above tensor to the reference configuration is:

$$ {}^{t}\underline{\underline{\mathbf{T}}}^{\sharp} = {}^{t}\tilde{\tau}^{ij}\ {}^{\circ}\underline{\tilde{\mathbf{g}}}_i\ {}^{\circ}\underline{\tilde{\mathbf{g}}}_j = \left[{}^{t}\underline{\underline{\mathbf{T}}}^{\sharp}\right]^{IJ}\ {}^{\circ}\underline{\mathbf{g}}_I\ {}^{\circ}\underline{\mathbf{g}}_J\ . \tag{3.15b} $$

We define a two-point representation of ${}^{t}\underline{\underline{\boldsymbol{\tau}}}$ as

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{P}}} = {}^{t}\tilde{\tau}^{ij}\ {}^{\circ}\underline{\tilde{\mathbf{g}}}_i\ {}^{t}\underline{\tilde{\mathbf{g}}}_j = \left[{}^{t}_{\circ}\underline{\underline{\mathbf{P}}}\right]^{Ij}\ {}^{\circ}\underline{\mathbf{g}}_I\ {}^{t}\underline{\mathbf{g}}_j\ . \tag{3.15c} $$

After some algebra,

$$ \begin{aligned} {}^{t}_{\circ}P^{Ij} &= {}^{t}\tilde{\tau}^{lm}\ \frac{\partial\,{}^{\circ}x^I}{\partial\theta^l}\ \frac{\partial\,{}^{t}x^j}{\partial\theta^m} = \left[{}^{t}\tau^{pq}\ \frac{\partial\theta^l}{\partial\,{}^{t}x^p}\ \frac{\partial\theta^m}{\partial\,{}^{t}x^q}\right]\ \frac{\partial\,{}^{\circ}x^I}{\partial\theta^l}\ \frac{\partial\,{}^{t}x^j}{\partial\theta^m} \\ &= {}^{t}\tau^{pj}\ \left({}^{t}_{\circ}X^{-1}\right)^{I}_{\ p}\ . \end{aligned} \tag{3.15d} $$

The second-order two-point tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{P}}}$ is the *first Piola-Kirchhoff stress tensor*. It is apparent from Eq. (3.15d) that it is a *non-symmetric* tensor.

We can write, due to the symmetry of the Kirchhoff stress tensor:

$$ P_\sigma = \int_{{}^{\circ}V} {}^{t}\tau^{ab}\ {}^{t}l_{ab}\ {}^{\circ}\mathrm{d}V = \int_{{}^{\circ}V} {}^{t}\tau^{ba}\ {}^{t}l_{ab}\ {}^{\circ}\mathrm{d}V\ . \tag{3.16a} $$

Hence, using Eq.(3.15d):

$$ P_\sigma = \int_{{}^{\circ}V} {}^{t}_{\circ}P^{Ba}\ {}^{t}_{\circ}X^{b}_{\ B}\ {}^{t}l_{ab}\ {}^{\circ}\mathrm{d}V\ . \tag{3.16b} $$

Using Eq. (2.111a) we have

$$ P_\sigma = \int_{{}^{\circ}V} {}^{t}_{\circ}P^{Ba}\ {}^{t}_{\circ}\dot{X}_{aB}\ {}^{\circ}\mathrm{d}V = \int_{{}^{\circ}V} {}^{t}_{\circ}\underline{\underline{\mathbf{P}}}\ \cdot\ \cdot\ {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{X}}}}\ {}^{\circ}\mathrm{d}V\ . \tag{3.17} $$

We can also write the above as (Malvern 1969):

$$ P_\sigma = \int_{{}^{\circ}V} {}^{t}_{\circ}\underline{\underline{\mathbf{P}}}^{T} : {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{X}}}}\ {}^{\circ}\mathrm{d}V\ . \tag{3.18} $$

The above equation defines the two-point tensors ${}^{t}_{\circ}\underline{\underline{\mathbf{P}}}^{T}$ and ${}^{t}_{\circ}\underline{\underline{\dot{\mathbf{X}}}}$ as *energy conjugates*.

We will not try to force a so-called "physical interpretation" of the first Piola-Kirchhoff stress tensor; instead we will regard it only as a useful mathematical tool.

3.3.3 The second Piola-Kirchhoff stress tensor

The pull-back configuration of ${}^t\underline{\underline{\boldsymbol{\tau}}}$ to the reference configuration is:

$$ {}^t\underline{\underline{\mathbf{T}}}^\sharp = \left[{}^t\tau^{ij} \left({}^t_\circ X^{-1} \right)^I_{\ i} \left({}^t_\circ X^{-1} \right)^J_{\ j} \right] {}^\circ\underline{\mathbf{g}}_I \, {}^\circ\underline{\mathbf{g}}_J \, . \tag{3.19} $$

The tensor ${}^t\underline{\underline{\mathbf{T}}}^\sharp$, defined by the above equation, is the *second Piola-Kirchhoff stress tensor* and it is a *symmetric tensor.*

Using Bathe's notation (Bathe 1996), we identify the second Piola-Kirchhoff stress tensor, corresponding to the t-configuration and referred to the configuration in $t = 0$ as ${}^t_\circ\underline{\underline{\mathbf{S}}}$.

Example 3.2. ◀◀◀◀◀

From Eqs. (3.15d) and (3.19) we get

$$ {}^t_\circ S^{IJ} = \left({}^t_\circ X^{-1} \right)^J_{\ j} \, {}^t_\circ P^{Ij} $$

that is to say,

$$ {}^t_\circ S^{IJ} = \left[{}^t\phi^* \left({}^t_\circ P^{Ij} \right) \right]^{IJ} \, . $$

◀◀◀◀◀

From Example 2.16:

$$ {}^t_\circ \dot{\varepsilon}_{AB} = \left[{}^t\phi^* \left({}^t d_{ab} \right) \right]_{AB} \tag{3.20a} $$

and since from Eq. (3.19),

$$ {}^t_\circ S^{AB} = \left[{}^t\phi^* \left({}^t\tau^{ab} \right) \right]^{AB} \tag{3.20b} $$

using Eqs. (3.11) and (2.88), we get

$$ {}^tP_\sigma = \int_{{}^\circ V} {}^t_\circ\underline{\underline{\mathbf{S}}} : {}^t_\circ\underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \, {}^\circ \mathrm{d}V \, . \tag{3.20c} $$

From Eq. (3.20c) we define the tensors ${}^t_\circ\underline{\underline{\mathbf{S}}}$ and ${}^t_\circ\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}$ to be *energy conjugate* (Atluri 1984).

Here, we will also not try to force a "physical interpretation" of the second Piola-Kirchhoff stress tensor.

An important point to be analyzed is the transformation of ${}^t_\circ\underline{\underline{\mathbf{S}}}$ under rigid-body rotations.

- Let us first consider the t-configuration of a certain body $\mathcal{B}$. At an arbitrary point P the Cauchy stress tensor is ${}^t\underline{\underline{\boldsymbol{\sigma}}}$.

- Let us now assume that we evolve from the t-configuration to a $(t+\Delta t)$-configuration imposing on $\mathcal{B}$ and on its external loads a rigid body rotation.

  At the point P we can write:

$$ {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{X}}} = {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \,, \tag{3.21a} $$

  and therefore,

$$ {}^{t+\Delta t}\mathrm{d}\underline{\mathbf{x}} = {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}} \,. \tag{3.21b} $$

  - For the external loads,

$$ {}^{t+\Delta t}\underline{\mathbf{t}} = {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{t}} \,, \tag{3.22a} $$

$$ {}^{t+\Delta t}\underline{\mathbf{b}} = {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{b}} \,. \tag{3.22b} $$

  - For a velocity vector,

$$ {}^{t+\Delta t}\underline{\mathbf{v}} = {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{v}} \,. \tag{3.22c} $$

  - For the external normal vector,

$$ {}^{t+\Delta t}\underline{\mathbf{n}} = {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{n}} \,. \tag{3.22d} $$

Example 3.3. ◀◀◀◀◀

For an arbitrary force vector ${}^{t}\underline{\mathbf{f}}$ (it can be a force per unit surface, per unit volume, etc.) and considering the evolution described above,

$$
\begin{aligned}
{}^{t+\Delta t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} &= {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} \\
&= {}^{t}\underline{\mathbf{f}} \cdot {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} \\
&= {}^{t}\underline{\mathbf{f}} \cdot {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{t}^{t+\Delta t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{v}}
\end{aligned}
$$

and since the rotation tensor is orthogonal,

$$ {}^{t+\Delta t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} = {}^{t}\underline{\mathbf{f}} \cdot {}^{t}\underline{\mathbf{v}} \,. $$

The above equation states the intuitive notion that a rigid-body rotation cannot affect the value of the deformation power performed by the external forces. ◀◀◀◀◀

At t we can write

$$ {}^{t}\underline{\mathbf{t}} = {}^{t}\underline{\mathbf{n}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \tag{3.23a} $$

and at $(t+\Delta t)$,

$$ {}^{t+\Delta t}\underline{\mathbf{t}} = {}^{t+\Delta t}\underline{\mathbf{n}} \cdot {}^{t+\Delta t}\underline{\underline{\boldsymbol{\sigma}}} \,. \tag{3.23b} $$

Introducing Eq. (3.22d) in the above,

$$ {}^{t+\Delta t}\underline{\mathbf{t}} = \left({}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{n}}\right) \cdot {}^{t+\Delta t}\underline{\underline{\boldsymbol{\sigma}}} \,. \tag{3.23c} $$

And with Eq. (2.28a) and (3.22a), we finally have

$$ {}^{t}\underline{\mathbf{t}} = {}^{t}\underline{\mathbf{n}} \cdot \left[{}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t+\Delta t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}\right] \,. \tag{3.23d} $$

For deriving the above equation, we used that ${}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{t}} = {}^{t}\underline{\mathbf{t}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T}$.

Hence,

$$ {}^{t+\Delta t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \,. \tag{3.23e} $$

The above equation indicates that the Cauchy stress tensor fulfills the criterion for objectivity under isometric transformations, established for Eulerian tensors in Sect. 2.12.2.

We define an arbitrary system $\{{}^{t}x^{a'}\}$ in the t-configuration and a system $\{{}^{t+\Delta t}x^{a}\}$ in the $(t+\Delta t)$-configuration. Hence, from Eq. (3.23e),

$$ {}^{t+\Delta t}\sigma^{a}{}_{b} = {}^{t+\Delta t}_{t}R^{a}{}_{c'}\; {}^{t}\sigma^{c'}{}_{d'}\; \left({}^{t+\Delta t}_{t}R^{T}\right)^{d'}{}_{b} \tag{3.24a} $$

and using Eq. (2.28c), we get

$$ {}^{t+\Delta t}\sigma^{a}{}_{b} = {}^{t+\Delta t}_{t}R^{a}{}_{c'}\; {}^{t}\sigma^{c'}{}_{d'}\; {}^{t+\Delta t}_{t}R^{l}{}_{m'}\; {}^{t+\Delta t}g_{lb}\; {}^{t}g^{m'd'} \tag{3.24b} $$

therefore,

$$ {}^{t+\Delta t}\sigma^{al} = {}^{t}\sigma^{c'm'}\; {}^{t+\Delta t}_{t}R^{a}{}_{c'}\; {}^{t+\Delta t}_{t}R^{l}{}_{m'} \,. \tag{3.24c} $$

It is easy to show that for the Kirchhoff stress tensor we can also write

$$ {}^{t+\Delta t}\underline{\underline{\boldsymbol{\tau}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\underline{\boldsymbol{\tau}}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \,. \tag{3.25} $$

From Eq. (3.19), we obtain

$$ {}^{t+\Delta t}_{\circ}S^{IJ} = {}^{t+\Delta t}\tau^{ij}\; \left({}^{t+\Delta t}_{\circ}X^{-1}\right)^{I}{}_{i}\; \left({}^{t+\Delta t}_{\circ}X^{-1}\right)^{J}{}_{j} \tag{3.26a} $$

but since,

$$ {}^{t+\Delta t}_{\circ}X^{a}{}_{A} = {}^{t+\Delta t}_{t}R^{a}{}_{a'}\; {}^{t}_{\circ}X^{a'}{}_{A} \tag{3.26b} $$

$$ \left({}^{t+\Delta t}_{\circ}X^{-1}\right)^{A}{}_{a} = \left({}^{t}_{\circ}X^{-1}\right)^{A}{}_{a'}\; \left({}^{t+\Delta t}_{t}R^{T}\right)^{a'}{}_{a} \tag{3.26c} $$

using Eqs. (3.24c) and (3.26c) in Eq. (3.26a), we get

$$ {}^{t+\Delta t}_{\circ}S^{IJ} = {}^{t}_{\circ}S^{IJ} \tag{3.27a}$$

therefore,

$$ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{S}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{S}}} \,. \tag{3.27b}$$

The above equation indicates that the second Piola-Kirchhoff stress tensor fulfills the criterion for objectivity under isometric transformations, established for Lagrangian tensors in Sect. 2.12.2.

3.3.4 A stress tensor energy conjugate to the time derivative of the Hencky strain tensor

In Sect. 2.8.5 we defined the logarithmic or Hencky strain tensor.

Let us now define, via a pull-back operation, the following stress tensor:

$$ {}^{t}\underline{\underline{\mathbf{\Gamma}}} = {}^{t}_{\circ}R^{*}\left({}^{t}\underline{\underline{\boldsymbol{\tau}}}\right) . \tag{3.28}$$

With the notation ${}^{t}_{\circ}R^{*}(\cdot)$ we define the pull-back of the components of the tensor $(\cdot)$ using the tensor ${}^{t}_{\circ}\underline{\underline{\mathbf{R}}}$ (Simo & Marsden 1984), that is to say, ${}^{t}\underline{\underline{\mathbf{\Gamma}}}$ is an *unrotated* representation of ${}^{t}\underline{\underline{\boldsymbol{\tau}}}$.

From the symmetry of ${}^{t}\underline{\underline{\boldsymbol{\tau}}}$, the above definition implies the symmetry of ${}^{t}\underline{\underline{\mathbf{\Gamma}}}$.

We will now demonstrate, following (Atluri 1984), that for an isotropic material ${}^{t}\underline{\underline{\mathbf{\Gamma}}}$ and ${}^{t}_{\circ}\underline{\underline{\dot{\mathbf{H}}}}$ are *energy conjugate*.

We can write Eq. (3.14) as

$$ {}^{t}P_{\sigma} = \int_{{}^{\circ}V} {}^{t}\Gamma^{AB} \left[{}^{t}_{\circ}R^{*}\left({}^{t}d_{ab}\right)\right]_{AB} \,{}^{\circ}\mathrm{d}V \tag{3.29a}$$

therefore,

$$ {}^{t}P_{\sigma} = \int_{{}^{\circ}V} {}^{t}\Gamma^{AB}\, {}^{t}_{\circ}R^{a}{}_{A}\, {}^{t}d_{ab}\, {}^{t}_{\circ}R^{b}{}_{B} \,{}^{\circ}\mathrm{d}V \,. \tag{3.29b}$$

From Eq. (2.28c),

$$ {}^{t}_{\circ}R^{a}{}_{A} = \left({}^{t}_{\circ}R^{T}\right)^{L}_{\;l}\, {}^{t}g^{al}\, {}^{\circ}g_{AL} \tag{3.29c}$$

and using the above, the integrand in Eq. (3.29b) is:

$$ {}^{t}\Gamma^{AB} \left({}^{t}_{\circ}R^{T}\right)^{L}_{\;l}\, {}^{t}g^{al}\, {}^{\circ}g_{AL}\, {}^{t}d_{ab}\, {}^{t}_{\circ}R^{b}{}_{B} \,. \tag{3.29d}$$

It is also easy to show that

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t}\underline{\underline{\mathbf{d}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} = \left[\left({}^{t}_{\circ}R^{T}\right)^{L}_{\;l}\, {}^{t}g^{al}\, {}^{\circ}g_{AL}\, {}^{t}d_{ab}\, {}^{t}_{\circ}R^{b}{}_{B}\right] {}^{\circ}\underline{\mathbf{g}}^{A}\, {}^{\circ}\underline{\mathbf{g}}^{B} \,. \tag{3.29e}$$

Hence, using Eq. (2.118b),

$$ {}^{t}P_{\sigma} = \int_{{}^{\circ}V} \frac{1}{2}\, {}^{t}\underline{\underline{\mathbf{\Gamma}}} : \left({}^{t}_{\circ}\underline{\underline{\dot{\mathbf{U}}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}}^{-1} + {}^{t}_{\circ}\underline{\underline{\mathbf{U}}}^{-1} \cdot {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{U}}}}\right) {}^{\circ}\mathrm{d}V \,. \tag{3.29f}$$

In order to simplify the algebra, in what follows we will work in a Cartesian system; $[A]$ will be the matrix formed with the Cartesian components of a second-order tensor $\underline{\underline{\mathbf{A}}}$.

From Eqs. (2.122a-2.122d), we get

$$[{}^t_{\circ}\dot{U}]\ [{}^t_{\circ}U]^{-1} \ = \ [{}^t_{\circ}R_L]\ [{}^t\dot{\Lambda}]\ [{}^t\Lambda]^{-1}\ [{}^t_{\circ}R_L]^T \ + \ [{}^t_{\circ}\Omega_L] \\ - [{}^t_{\circ}R_L]\ [{}^t\Lambda]\ [{}^t_{\circ}R_L]^T\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t_{\circ}R_L]^T \tag{3.30a}$$

and

$$[{}^t_{\circ}U]^{-1}\ [{}^t_{\circ}\dot{U}] \ = \ [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t\dot{\Lambda}]\ [{}^t_{\circ}R_L]^T \\ + [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t_{\circ}R_L]^T\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}R_L]\ [{}^t\Lambda]\ [{}^t_{\circ}R_L]^T \ - \ [{}^t_{\circ}\Omega_L] \tag{3.30b}$$

it follows from the above two equations that

$$\frac{1}{2}\left\{[{}^t_{\circ}\dot{U}]\ [{}^t_{\circ}U]^{-1} \ + \ [{}^t_{\circ}U]^{-1}\ [{}^t_{\circ}\dot{U}]\right\} \ = \ [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t\dot{\Lambda}]\ [{}^t_{\circ}R_L]^T \\ + \frac{1}{2}\ [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t_{\circ}R_L]^T\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}R_L]\ [{}^t\Lambda]\ [{}^t_{\circ}R_L]^T \\ - \frac{1}{2}\ [{}^t_{\circ}R_L]\ [{}^t\Lambda]\ [{}^t_{\circ}R_L]^T\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t_{\circ}R_L]^T\ , \tag{3.30c}$$

and using once again Eqs. (2.122a-2.122d), we get

$$\frac{1}{2}\left\{[{}^t_{\circ}\dot{U}]\ [{}^t_{\circ}U]^{-1} \ + \ [{}^t_{\circ}U]^{-1}\ [{}^t_{\circ}\dot{U}]\right\} \ = \ [{}^t_{\circ}R_L]\ [{}^t\Lambda]^{-1}\ [{}^t\dot{\Lambda}]\ [{}^t_{\circ}R_L]^T \\ + \frac{1}{2}\ [{}^t_{\circ}U]^{-1}\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}U] \ - \ \frac{1}{2}\ [{}^t_{\circ}U]\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}U]^{-1}\ . \tag{3.30d}$$

Using the result in Example 2.17, we can write

$$\frac{1}{2}\left\{[{}^t_{\circ}\dot{U}]\ [{}^t_{\circ}U]^{-1} \ + \ [{}^t_{\circ}U]^{-1}\ [{}^t_{\circ}\dot{U}]\right\} \ = \ [{}^t_{\circ}\dot{H}] \ - \ [{}^t_{\circ}\Omega_L]\ [ln{}^t_{\circ}U] \\ + [ln{}^t_{\circ}U]\ [{}^t_{\circ}\Omega_L] + \frac{1}{2}\ [{}^t_{\circ}U]^{-1}\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}U] \ - \ \frac{1}{2}\ [{}^t_{\circ}U]\ [{}^t_{\circ}\Omega_L]\ [{}^t_{\circ}U]^{-1}\ . \tag{3.30e}$$

Using the above in Eq. (3.29f) and working with the matrix components,

$$ {}^tP_\sigma \ = \ \int_{{}^\circ V} [{}^t\Gamma]_{\alpha\beta}\ \Big\{[{}^t_{\circ}\dot{H}]_{\alpha\beta} \ - \ [{}^t_{\circ}\Omega_L]_{\alpha\gamma}\ [ln{}^t_{\circ}U]_{\gamma\beta} \\ + [ln{}^t_{\circ}U]_{\alpha\gamma}\ [{}^t_{\circ}\Omega_L]_{\gamma\beta} + \frac{1}{2}\ [{}^t_{\circ}U^{-1}]_{\alpha\gamma}\ [{}^t_{\circ}\Omega_L]_{\gamma\delta}\ [{}^t_{\circ}U]_{\delta\beta} \\ - \frac{1}{2}\ [{}^t_{\circ}U]_{\alpha\gamma}\ [{}^t_{\circ}\Omega_L]_{\gamma\delta}\ [{}^t_{\circ}U^{-1}]_{\delta\beta}\ \Big\}\ {}^\circ\mathrm{d}V\ . \tag{3.31a}$$

Since $[{}^t\Gamma]$ and $[{}^t_{\circ}U]$ are symmetric and $[{}^t\Omega_L]$ is skew-symmetric, we can rewrite the above equation as

$$
\begin{aligned}
{}^{t}P_{\sigma} \;=\; & \int_{{}^{\circ}V} [{}^{t}\varGamma]_{\alpha\beta}\; [{}^{t}_{\circ}\dot{H}]_{\alpha\beta}\; {}^{\circ}\mathrm{d}V \\
& - \int_{{}^{\circ}V} \Big\{ \big[[{}^{t}\varGamma]\,[{}^{t}_{\circ}H]\big]_{\beta\gamma} \;-\; \big[[{}^{t}_{\circ}H]\,[{}^{t}\varGamma]\big]_{\beta\gamma} \Big\}\; [{}^{t}_{\circ}\varOmega_L]_{\beta\gamma}\; {}^{\circ}\mathrm{d}V \\
+ \frac{1}{2} & \int_{{}^{\circ}V} \Big\{ \big[[{}^{t}_{\circ}U]^{-1}\,[{}^{t}\varGamma]\,[{}^{t}_{\circ}U]\big]_{\gamma\delta} \;-\; \big[[{}^{t}_{\circ}U]\,[{}^{t}\varGamma]\,[{}^{t}_{\circ}U]^{-1}\big]_{\gamma\delta} \Big\}\; [{}^{t}_{\circ}\varOmega_L]_{\gamma\delta}\; {}^{\circ}\mathrm{d}V\,.
\end{aligned}
\tag{3.31b}
$$

We will show in Chap. 5 that for *isotropic materials* the Eulerian tensors ${}^{t}\underline{\underline{\boldsymbol{\sigma}}}$ (${}^{t}\underline{\underline{\boldsymbol{\tau}}}$) and ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$ have coincident eigenvectors (they are coaxial).

Taking into account that

$$
{}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \;=\; {}^{t}_{\circ}R^{*}({}^{t}_{\circ}\underline{\underline{\mathbf{V}}}) \tag{3.32}
$$

and the definition of ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$ in Eq. (3.28) we conclude that the Lagrangian tensors ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$, ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ and ${}^{t}_{\circ}\underline{\underline{\mathbf{H}}}$ are also coaxial for *isotropic materials*.

Obviously, the coaxiality of ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$ and ${}^{t}_{\circ}\underline{\underline{\mathbf{H}}}$ implies that

$$
[{}^{t}\varGamma]\,[{}^{t}_{\circ}H] \;-\; [{}^{t}_{\circ}H]\,[{}^{t}\varGamma] \;=\; [0] \tag{3.33a}
$$

and the coaxiality of ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$ and ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ implies that

$$
[{}^{t}_{\circ}U]^{-1}\,[{}^{t}\varGamma]\,[{}^{t}_{\circ}U] \;-\; [{}^{t}_{\circ}U]\,[{}^{t}\varGamma]\,[{}^{t}_{\circ}U]^{-1} \;=\; [0]\,. \tag{3.33b}
$$

Finally, for *isotropic materials*,

$$
{}^{t}P_{\sigma}^{Isot.Mat.} \;=\; \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\Gamma}}} : {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{H}}}}\; {}^{\circ}\mathrm{d}V \tag{3.34}
$$

and therefore in this case ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$ and ${}^{t}_{\circ}\dot{\underline{\underline{\mathbf{H}}}}$ are energy conjugates.

3.4 Objective stress rates

In Sect. 2.14.1 we show that the adequate tool for deriving objective rates of Eulerian tensors is the Lie derivative.

The Lie derivative of the Cauchy stress tensor is:

$$
\left({}^{t}\overset{\circ}{\underline{\underline{\boldsymbol{\sigma}}}}\right)^{ab} \;=\; \left[L_{{}^{t}\underline{\mathbf{v}}}({}^{t}\underline{\underline{\boldsymbol{\sigma}}})\,\right]^{ab} = {}^{t}\dot{\sigma}^{ab} \;-\; {}^{t}\sigma^{cb}\;{}^{t}l^{a}_{\;\;c} \;-\; {}^{t}\sigma^{ac}\;{}^{t}l^{b}_{\;\;c} \tag{3.35}
$$

the above stress rate is known as *Oldroyd stress rate* (Marsden & Hughes 1983).

Example 3.4. ◀◀◀◀◀◀

To derive the expression of *Oldroyd's stress rate* we start from Eq. (2.129),

$$\left(L_{{}^t\underline{\mathbf{v}}}\,{}^t\underline{\underline{\boldsymbol{\sigma}}}\right)^{ab} = \frac{\partial^t\sigma^{ab}}{\partial t} + \frac{\partial^t\sigma^{ab}}{\partial^t x^p}\;{}^tv^p - \frac{\partial^t v^a}{\partial^\circ x^P}\;\left({}^t_\circ X^{-1}\right)^P{}_p\;{}^t\sigma^{pb} - \frac{\partial^t v^b}{\partial^\circ x^P}\;\left({}^t_\circ X^{-1}\right)^P{}_p\;{}^t\sigma^{ap}\,.$$

From Eq. (2.109a)

$${}^t_\circ\dot{\underline{\underline{\mathbf{X}}}} = \left[\frac{\partial^t v^a}{\partial^\circ x^A} + {}^t_\circ X^p{}_A\;{}^t\Gamma^a_{pl}\;{}^tv^l\right]{}^t\underline{\mathbf{g}}_a\;{}^\circ\underline{\mathbf{g}}^A\,,$$

and Eq. (2.111b)

$${}^t_\circ\dot{\underline{\underline{\mathbf{X}}}} = {}^t\underline{\underline{\mathbf{l}}}\cdot{}^t_\circ\underline{\underline{\mathbf{X}}}\,,$$

we get,

$$\frac{\partial^t v^a}{\partial^\circ x^A} = {}^tl^a{}_m\;{}^t_\circ X^m{}_A - {}^t\Gamma^a_{rs}\;{}^tv^s\;{}^t_\circ X^r{}_A\,.$$

Then, from

$${}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^t\sigma^{ab}\;{}^t\underline{\mathbf{g}}_a\;{}^t\underline{\mathbf{g}}_b\,,$$

we get,

$${}^t\dot{\underline{\underline{\boldsymbol{\sigma}}}} = \left[\frac{\partial^t\sigma^{ab}}{\partial t} + \frac{\partial^t\sigma^{ab}}{\partial^t x^l}\;{}^tv^l + {}^t\sigma^{mb}\;{}^t\Gamma^a_{ml}\;{}^tv^l + {}^t\sigma^{am}\;{}^t\Gamma^b_{ml}\;{}^tv^l\right]\;{}^t\underline{\mathbf{g}}_a\;{}^t\underline{\mathbf{g}}_b\,.$$

Replacing in the first equation, after some algebra, we finally get

$$\left(L_{{}^t\underline{\mathbf{v}}}\;{}^t\underline{\underline{\boldsymbol{\sigma}}}\right)^{ab} = {}^t\dot{\sigma}^{ab} - {}^tl^a{}_p\;{}^t\sigma^{pb} - {}^tl^b{}_p\;{}^t\sigma^{ap}\,.$$

◀◀◀◀◀◀

- The Lie derivative of the Kirchhoff stress tensor is:

$$\left[{}^t\overset{\circ}{\underline{\underline{\boldsymbol{\tau}}}}\right]^{ab} = \left[\,L_{{}^t\underline{\mathbf{v}}}({}^t\underline{\underline{\boldsymbol{\tau}}})\right]^{ab} = {}^t\dot{\tau}^{ab} - {}^t\tau^{cb}\;{}^tl^a{}_c - {}^t\tau^{ac}\;{}^tl^b{}_c \tag{3.36}$$

the above rate is known as the *Truesdell stress rate* (Marsden & Hughes 1983).

Example 3.5. ◀◀◀◀◀◀

From Eq. (3.19),

$${}^t\tau^{ij} = {}^t_\circ S^{IJ}\;{}^t_\circ X^i{}_I\;{}^t_\circ X^j{}_J\,,$$

hence,

$$\frac{\partial\, {}^t\tau^{ij}}{\partial t} + \frac{\partial\, {}^t\tau^{ij}}{\partial\, {}^tx^p}\, {}^tv^p = {}^t_\circ\dot{S}^{IJ}\ {}^t_\circ X^i{}_I\ {}^t_\circ X^j_J + {}^t_\circ S^{IJ}\ \frac{\partial\, {}^tv^i}{\partial\, {}^\circ x^I}\, {}^t_\circ X^j{}_J$$
$$+\, {}^t_\circ S^{IJ}\ {}^t_\circ X^i{}_I\ \frac{\partial\, {}^tv^j}{\partial\, {}^\circ x^J}$$

but

$$\left[{}^t\phi_*\ ({}^t_\circ\dot{S}^{IJ})\right]^{ij} = {}^t_\circ\dot{S}^{IJ}\ {}^t_\circ X^i{}_I\ {}^t_\circ X^j_J\ .$$

Therefore,

$$\left[{}^t\phi_*\ ({}^t_\circ\dot{S}^{IJ})\right]^{ij} = \frac{\partial\, {}^t\tau^{ij}}{\partial t} + \frac{\partial\, {}^t\tau^{ij}}{\partial\, {}^tx^p}\, {}^tv^p$$
$$-\, {}^t\tau^{lm}\ ({}^t_\circ X^{-1})^I{}_l\ ({}^t_\circ X^{-1})^J{}_m\ \frac{\partial\, {}^tv^i}{\partial\, {}^\circ x^I}\, {}^t_\circ X^j{}_J$$
$$-\, {}^t\tau^{lm}\ ({}^t_\circ X^{-1})^I{}_l\ ({}^t_\circ X^{-1})^J{}_m\ \frac{\partial\, {}^tv^j}{\partial\, {}^\circ x^J}\, {}^t_\circ X^i{}_I\ .$$

Using algebra along the lines of Example 3.4 and Eq.(3.36) we finally get,

$${}^t\overset{\circ}{\tau}{}^{ij} = \left[{}^t\phi_*\ ({}^t_\circ\dot{S}^{IJ})\right]^{ij}\ .$$

The above example shows that the necessary and sufficient condition for the second Piola-Kirchhoff stress tensor to remain constant is that the Truesdell stress rate is zero (Eringen 1967).

We can also perform pull-back and push-forward operations using the rotation tensor ${}^t_\circ\underline{\underline{\mathbf{R}}}$ (Simo & Marsden 1984). Let us consider an *arbitrary Eulerian stress tensor* ${}^t\underline{\underline{\mathbf{t}}}$ (e.g. ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ or ${}^t\underline{\underline{\boldsymbol{\tau}}}$), and perform on its components a ${}^t_\circ R$-pull-back:

$$\left[{}^t_\circ R^*({}^t\underline{\underline{\mathbf{t}}})\right]^{AB} = {}^tt^{ab}\ ({}^t_\circ R^T)^A_{\ a}\ ({}^t_\circ R^T)^B_{\ b}\ . \tag{3.37a}$$

In order to simplify our calculations we will now work in a Cartesian system; hence,

$$\left[{}^t_\circ R^*({}^t\underline{\underline{\mathbf{t}}})\right] = [{}^t_\circ R]^T\ [{}^tt]\ [{}^t_\circ R]\ . \tag{3.37b}$$

Taking into account that $\frac{\mathrm{d}}{\mathrm{d}t}\left\{[R]^T\right\} = \left\{\frac{\mathrm{d}}{\mathrm{d}t}[R]\right\}^T$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[{}^t_\circ R^*({}^t\underline{\underline{\mathbf{t}}})\right] = [{}^t_\circ R]^T\ [{}^t\dot{t}]\ [{}^t_\circ R] + [{}^t_\circ\dot{R}]^T\ [{}^tt]\ [{}^t_\circ R] + [{}^t_\circ R]^T\ [{}^tt]\ [{}^t_\circ\dot{R}] \tag{3.37c}$$

and using Eqs. (2.115a) and (2.116a) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[{}^t_\circ R^*({}^t\underline{\underline{\mathbf{t}}})\right] = [{}^t_\circ R]^T\ [{}^t\dot{t}]\ [{}^t_\circ R] - [{}^t_\circ R]^T\ [{}^t\Omega_R]\ [{}^tt]\ [{}^t_\circ R]$$
$$+\, [{}^t_\circ R]^T\ [{}^tt]\ [{}^t\Omega_R]\ [{}^t_\circ R]\ . \tag{3.37d}$$

Since,

$$\left[L_{{}^t_\circ\underline{\underline{\mathbf{R}}}}({}^t\underline{\underline{\mathbf{t}}})\right] = [{}^t_\circ R]\,\frac{\mathrm{d}}{\mathrm{d}t}\left[{}^t_\circ R^*({}^t\underline{\underline{\mathbf{t}}})\right]\,[{}^t_\circ R]^T \tag{3.37e}$$

we finally arrive at

$$\left[L_{{}^t_\circ\underline{\underline{\mathbf{R}}}}({}^t\underline{\underline{\mathbf{t}}})\right] = [{}^t\dot{t}] - [{}^t\Omega_R]\,[{}^t t] + [{}^t t]\,[{}^t\Omega_R] \tag{3.37f}$$

that in an arbitrary spatial coordinate system leads to

$$\left[L_{{}^t_\circ\underline{\underline{\mathbf{R}}}}({}^t\underline{\underline{\mathbf{t}}})\right]^{ab} = {}^t\dot{t}^{ab} - {}^t\Omega_{R\,c}^{\;a}\,{}^t t^{cb} + {}^t t^{ac}\,{}^t\Omega_{R\,c}^{\;b}\,. \tag{3.38}$$

The above Lie derivative is the well-known *Green-Naghdi stress rate* (Dienes 1979, Marsden & Hughes 1983, Pinsky, Ortiz & Pister 1983, Simo & Pister 1984, Cheng & Tsui 1990). From its derivation it is apparent that the Green-Naghdi stress rate is *objective under isometric transformations*, therefore it is known as a *corotational stress rate.*

In the case of the Kirchhoff stress tensor, its Green-Naghdi rate is the ${}^t_\circ R$-push-forward of the rate of ${}^t\underline{\underline{\mathbf{\Gamma}}}$.

If as a reference configuration we use the t-configuration (Dienes 1979), we will fulfill Eq. (2.118d), that is to say,

$${}^{\mathbf{t}}\underline{\underline{\boldsymbol{\omega}}} = {}^t\underline{\underline{\boldsymbol{\Omega}}}_R\,. \tag{3.39}$$

Using the above in Eq. (3.38) we obtain the *Jaumann stress rate* (Truesdell & Noll 1965).

It is important to realize that (Dienes 1979):

- The Jaumann stress rate is only coincident with the corotational stress rate when ${}^t_\circ\underline{\underline{\mathbf{U}}} \approx {}^\circ\underline{\underline{\mathbf{g}}}$.
- When formulating a corotational constitutive relation, we will only be able to use the Jaumann stress rate when the spatial and reference configurations are coincident.

4

Balance principles

In this chapter we are going to present a set of basic equations (balance of mass, momentum, angular momentum and energy) that govern the behavior of the continuous media in the framework of Newtonian mechanics.

We are going to present these basic principles in an integral form and also in the form of partial differential equations (localized form). On presenting the basic principles we are going to use both, the Eulerian (spatial) and Lagrangian (material) descriptions of motion.

Some reference books for this chapter are: (Truesdell & Toupin 1960, Fung 1965, Eringen 1967, Malvern 1969, Slaterry 1972, Oden & Reddy 1976, Marsden & Hughes 1983, Panton 1984, Lubliner 1985, Fung & Tong 2001).

4.1 Reynolds' transport theorem

We begin this chapter by presenting Reynolds' transport theorem; which will be used in what follows as a tool for calculating material derivatives of integrals defined in a spatial domain.

Let us define in the spatial configuration of a continuum body $\mathcal{B}$ an arbitrary coordinate system $\{{}^t x^i \ , i \ = \ 1, 2, 3\}$ and let us assume a continuous Eulerian tensor field ${}^{\mathbf{t}}\boldsymbol{\psi}({}^t x^i \ , t)$ to be a single-valued function of the coordinates $\{{}^t x^i\}$ and of time t. Also, we define in the reference configuration an arbitrary coordinate system $\{{}^\circ x^I \ , I \ = \ 1, 2, 3\}$.

We define ${}^t V$ as a volume in the spatial configuration and

$$\frac{D}{Dt} \int_{{}^t V} {}^t\boldsymbol{\psi} \ ({}^t x_i, t) \ {}^t \mathrm{d}V \tag{4.1}$$

to be the *material time derivative of a spatial volume integral*; that is to say, Eq. (4.1) measures the rate of change of the total amount of the tensorial property ${}^{\mathbf{t}}\boldsymbol{\psi}$ carried by the particles that at time t are inside the volume ${}^t V$.

Using Eq.(2.31) we can write,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\,({}^tx^i,t)\;{}^t\mathrm{d}V \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{{}^\circ V} {}^t\boldsymbol{\psi}\,({}^\circ x^I,t)\;{}^tJ\;{}^\circ\mathrm{d}V \tag{4.2a}$$

hence,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\,({}^tx^i,t)\,{}^t\mathrm{d}V \;=\; \int_{{}^\circ V} \frac{\mathrm{d}{}^t\boldsymbol{\psi}\,({}^\circ x^I,t)}{\mathrm{d}t}\,{}^tJ\;{}^\circ\mathrm{d}V + \int_{{}^\circ V} {}^t\boldsymbol{\psi}\,({}^\circ x^I,t)\,\frac{\mathrm{d}{}^tJ}{\mathrm{d}t}\,{}^\circ\mathrm{d}V\;. \tag{4.2b}$$

Example 4.1. ◀◀◀◀◀

Working in Cartesian coordinates we can write, from Eq. (2.34e) (Fung 1965):

$${}^tJ \;=\; e_{\alpha\beta\gamma}\;{}^t_\circ X_{\alpha 1}\;{}^t_\circ X_{\beta 2}\;{}^t_\circ X_{\gamma 3}\;.$$

Using Eqs.(2.111a-2.111b) we can calculate the time rate of ${}^tJ$:

$$\begin{aligned}{}^t\dot{J} \;=\; e_{\alpha\beta\gamma}\;[{}^tl_{\alpha\epsilon}\;{}^t_\circ X_{\epsilon 1}\;{}^t_\circ X_{\beta 2}\;{}^t_\circ X_{\gamma 3} \;+\; {}^t_\circ X_{\alpha 1}\;{}^tl_{\beta\epsilon}\;{}^t_\circ X_{\epsilon 2}\;{}^t_\circ X_{\gamma 3}\\ +\; {}^t_\circ X_{\alpha 1}\;{}^t_\circ X_{\beta 2}\;{}^tl_{\gamma\epsilon}\;{}^t_\circ X_{\epsilon 3}]\;.\end{aligned}$$

After some algebra, the reader can easily verify that,

$${}^t\dot{J} \;=\; {}^tl_{\epsilon\epsilon}\;e_{\alpha\beta\gamma}\;{}^t_\circ X_{\alpha 1}\;{}^t_\circ X_{\beta 2}\;{}^t_\circ X_{\gamma 3}\;.$$

Generalizing the above for any set of curvilinear coordinates in the Euclidean space we can write,

$${}^t\dot{J} \;=\; (\underline{\boldsymbol{\nabla}}\;\cdot\;{}^t\underline{\mathbf{v}})\;{}^tJ$$

where ${}^t\underline{\mathbf{v}}$ is the velocity vector at the t-configuration.

A proof of the above result in general curvilinear coordinates can be found in (Marsden & Hughes 1983). ◀◀◀◀◀

Using the result of Example 4.1 in Eq. (4.2b) we obtain,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\,({}^tx^i,t)\;{}^t\mathrm{d}V \;= \tag{4.3a}$$

$$\int_{{}^\circ V}\left[\frac{\mathrm{d}{}^t\boldsymbol{\psi}\,({}^\circ x^I,t)}{\mathrm{d}t} \;+\; {}^t\boldsymbol{\psi}\,({}^\circ x^I,t)\;(\underline{\boldsymbol{\nabla}}\;\cdot\;{}^t\underline{\mathbf{v}})\right]\;{}^tJ\;{}^\circ\mathrm{d}V\;.$$

Returning to the t-configuration we obtain,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\,({}^tx^i,t)\;{}^t\mathrm{d}V \;= \tag{4.3b}$$

$$\int_{{}^tV}\left[\frac{D{}^t\boldsymbol{\psi}\,({}^tx^i,t)}{Dt} \;+\; {}^t\boldsymbol{\psi}\,({}^tx^i,t)\;(\underline{\boldsymbol{\nabla}}\;\cdot\;{}^t\underline{\mathbf{v}})\right]\;{}^t\mathrm{d}V\;.$$

Equation (4.3b) is one way of expressing Reynolds' transport theorem. In this Section, we will also discuss other expressions for this theorem.

Using Eq. (2.20b) we can rewrite Eq. (4.3b) as,

$$\begin{aligned}\frac{D}{Dt}\int_{{}^tV} {}^{\mathbf{t}}\boldsymbol{\psi}\ ({}^tx^i,t)\ {}^t\mathrm{d}V \ &= \int_{{}^tV} [\frac{\partial {}^{\mathbf{t}}\boldsymbol{\psi}\ ({}^tx^i,t)}{\partial t} \\ &+ {}^t\underline{\mathbf{v}}\cdot\underline{\boldsymbol{\nabla}}{}^{\mathbf{t}}\boldsymbol{\psi}\ ({}^tx^i,t) + {}^{\mathbf{t}}\boldsymbol{\psi}\ ({}^tx^i,t)(\underline{\boldsymbol{\nabla}}\ \cdot\ {}^t\underline{\mathbf{v}})]\ \ {}^t\mathrm{d}V\ .\end{aligned} \tag{4.3c}$$

Example 4.2. ◀◀◀◀◀

Let us assume a general tensor field:

$${}^t\boldsymbol{\psi} \ = \ {}^t\psi^{a\ldots b}{}_{c\ldots d}\ {}^t\underline{\mathbf{g}}_a \ldots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \ldots {}^t\underline{\mathbf{g}}^d$$

where the ${}^t\underline{\mathbf{g}}_i$ are the covariant base vectors of the coordinate system $\{{}^tx^i\}$. The velocity vector field can be written as,

$${}^t\underline{\mathbf{v}} \ = \ {}^tv^s\ {}^t\underline{\mathbf{g}}_s$$

therefore we can write, using Eq.(A.59),

$$\begin{aligned}{}^t\underline{\mathbf{v}}\cdot\left(\underline{\boldsymbol{\nabla}}{}^{\mathbf{t}}\boldsymbol{\psi}\ \right) &= \ {}^tv^s\ {}^t\underline{\mathbf{g}}_s\ \cdot\ {}^t\psi^{a\ldots b}{}_{c\ldots d}\,|_n\ \ {}^t\underline{\mathbf{g}}^n\ {}^t\underline{\mathbf{g}}_a \ldots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \ldots {}^t\underline{\mathbf{g}}^d \\ &= \ {}^tv^s\ {}^t\psi^{a\ldots b}{}_{c\ldots d}\,|_s\ \ {}^t\underline{\mathbf{g}}_a \ldots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \ldots {}^t\underline{\mathbf{g}}^d\ .\end{aligned}$$

Also, using Eq.(A.64),

$${}^{\mathbf{t}}\boldsymbol{\psi}(\underline{\boldsymbol{\nabla}}\ \cdot\ {}^t\underline{\mathbf{v}}) \ = \ {}^t\psi^{a\ldots b}{}_{c\ldots d}\ {}^tv^s|_s\ {}^t\underline{\mathbf{g}}_a \ldots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \ldots {}^t\underline{\mathbf{g}}^d \qquad .$$

Finally, using Eq. (A.62b),

$$\begin{aligned}\underline{\boldsymbol{\nabla}}\ \cdot\ ({}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi}) &= \left({}^tv^s\ \ {}^t\psi^{a\ldots b}{}_{c\ldots d}\right)|_s\ \ {}^t\underline{\mathbf{g}}_a \ldots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \ldots {}^t\underline{\mathbf{g}}^d \\ &= \left({}^tv^s\ \ {}^t\psi^{a\ldots b}{}_{c\ldots d}\,|_s + \ {}^t\psi^{a\ldots b}{}_{c\ldots d}\ {}^tv^s\,|_s\right)\ {}^t\underline{\mathbf{g}}_a \ldots {}^t\underline{\mathbf{g}}_b\ {}^t\underline{\mathbf{g}}^c \ldots {}^t\underline{\mathbf{g}}^d\ .\end{aligned}$$

From the above equations we get,

$${}^t\underline{\mathbf{v}}\ \cdot\ (\underline{\boldsymbol{\nabla}}\ {}^{\mathbf{t}}\boldsymbol{\psi})\ +\ {}^{\mathbf{t}}\boldsymbol{\psi}\ (\underline{\boldsymbol{\nabla}}\ \cdot\ {}^t\underline{\mathbf{v}})\ =\ \underline{\boldsymbol{\nabla}}\ \cdot\ ({}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi})\ .$$

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Using the above result we can express Reynolds' transport theorem as,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\,({}^tx^i,t)\,{}^t\mathrm{d}V \;=\; \int_{{}^tV}\left[\frac{\partial\,{}^t\boldsymbol{\psi}\,({}^tx^i,t)}{\partial t} + \underline{\boldsymbol{\nabla}}\cdot({}^t\underline{\mathbf{v}}\,{}^t\boldsymbol{\psi}({}^tx^i,t))\right]{}^t\mathrm{d}V\,. \tag{4.4}$$

The *generalized Gauss' theorem* can be stated as (Malvern 1969):

$$\int_{{}^tV}\underline{\boldsymbol{\nabla}}\cdot{}^t\boldsymbol{\psi}\;{}^t\mathrm{d}V \;=\; \int_{{}^tS}{}^t\underline{\mathbf{n}}\cdot{}^t\boldsymbol{\psi}\;\;{}^tdS \tag{4.5}$$

where ${}^tS$ is the closed surface that bounds the volume ${}^tV$ and ${}^t\underline{\mathbf{n}}$ is the surface's outer normal vector.

Using Gauss' theorem in Eq. (4.4) and rearranging terms, we get the following expression of Reynolds' transport theorem:

$$\int_{{}^tV}\frac{\partial\,{}^t\boldsymbol{\psi}}{\partial t}\,{}^t\mathrm{d}V \;=\; \frac{D}{Dt}\int_{{}^tV}{}^t\boldsymbol{\psi}\,{}^t\mathrm{d}V \;-\; \int_{{}^tS}{}^t\underline{\mathbf{n}}\cdot{}^t\underline{\mathbf{v}}\,{}^t\boldsymbol{\psi}\,{}^t\mathrm{d}S\,. \tag{4.6}$$

Following (Malvern 1969) we can state:

$$\left\{\begin{array}{c}\text{Rate of increase}\\ \text{of the total amount}\\ \text{of } {}^t\boldsymbol{\psi} \text{ inside}\\ \text{a volume } {}^tV\\ \text{in the spatial}\\ \text{configuration}\end{array}\right\} = \left\{\begin{array}{c}\text{Rate of increase}\\ \text{of the total amount}\\ \text{of } {}^t\boldsymbol{\psi} \text{ possessed}\\ \text{by the material}\\ \text{instantaneously}\\ \text{inside the volume } {}^tV\end{array}\right\} - \left\{\begin{array}{c}\text{Net rate of outward}\\ \text{flux of } {}^t\boldsymbol{\psi} \text{ carried}\\ \text{by the material}\\ \text{transport through}\\ \text{the closed surface } {}^tS\end{array}\right\}$$

The volume ${}^tV$ is usually called the *control volume* and the surface ${}^tS$ is usually called the *control surface*.

4.1.1 Generalized Reynolds' transport theorem

In the previous section, for deriving Reynolds' transport theorem we considered a *material volume* ${}^tV$ bounded by a *material surface* ${}^tS$. In this section, we are going to generalize the previous derivation considering in the spatial t-configuration an arbitrary volume $\Omega(t)$ bounded by a surface $\sigma(t)$ that moves with an arbitrary velocity field ${}^t\underline{\mathbf{w}}$.

We can define inside the t-configuration of the body $\mathcal{B}$ a surface,

$${}^tf\;({}^tx^i,t) \;=\; 0\,. \tag{4.7}$$

If the surface moves with the particles instantaneously on it, we say that it is a *material surface*.

The *Lagrange criterion* states (Truesdell & Toupin 1960) that the necessary and sufficient condition for the above-defined surface to be material is that its material time derivative is zero; using Eq.(2.20b),

$${}^t\dot{f} \;=\; \frac{\partial\,{}^tf}{\partial t} + \frac{\partial\,{}^tf}{\partial\,{}^tx^k}\,{}^tv^k \;=\; 0\,. \tag{4.8}$$

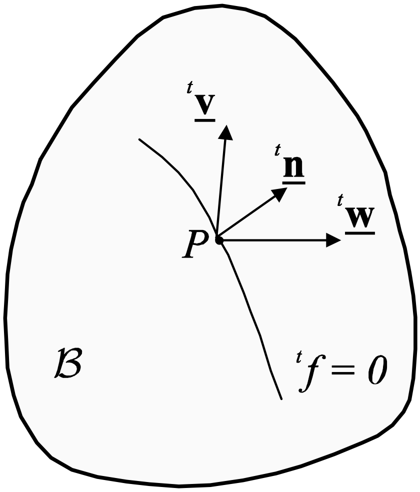

Fig. 4.1. Body $\mathcal{B}$ with the surface $^tf\left(^tx^i, t\right) = 0$

To prove the Lagrange criterion, let us consider the following general case, In Fig. 4.1, at a point P on $^tf = 0$ we define:
- The unit vector $^t\underline{\mathbf{n}}$ normal to the surface.
- The material velocity $^t\underline{\mathbf{v}}$ of the particle instantaneously at P.
- The velocity $^t\underline{\mathbf{w}}$ of the surface.

The condition that the point P (not the particle instantaneously at P) remains on $^tf = 0$ when the surface moves is given by,

$$\frac{\partial\, ^tf}{\partial t} + \frac{\partial\, ^tf}{\partial\, ^tx^k}\, ^tw^k = 0 \,. \tag{4.9a}$$

From geometrical considerations,

$$^t\underline{\mathbf{n}} = \frac{\frac{\partial\, ^tf}{\partial\, ^tx^k}\, ^t\underline{\mathbf{g}}^k}{\sqrt{\frac{\partial\, ^tf}{\partial\, ^tx^l}\, \frac{\partial\, ^tf}{\partial\, ^tx^m}\, ^tg^{lm}}} \tag{4.9b}$$

and we can write,

$$^t\underline{\mathbf{w}} = {^tw^p}\, ^t\underline{\mathbf{g}}_p \,. \tag{4.9c}$$

Hence,

$$^tw_n = {^t\underline{\mathbf{w}}} \cdot {^t\underline{\mathbf{n}}} = \frac{^tw^k\, \frac{\partial\, ^tf}{\partial\, ^tx^k}}{\sqrt{\frac{\partial\, ^tf}{\partial\, ^tx^l}\, \frac{\partial\, ^tf}{\partial\, ^tx^m}\, ^tg^{lm}}} \tag{4.9d}$$

and using in the above Eq.(4.9a), we get

$$^tw_n = -\frac{\frac{\partial\, ^tf}{\partial t}}{\sqrt{\frac{\partial\, ^tf}{\partial\, ^tx^l}\, \frac{\partial\, ^tf}{\partial\, ^tx^m}\, ^tg^{lm}}} \,. \tag{4.9e}$$

We can also define, using the material velocity of the particle instantaneously at P,

$$ {}^{t}v_n = {}^{t}\underline{\mathbf{v}} \cdot {}^{t}\underline{\mathbf{n}} = \frac{{}^{t}v^k \, \frac{\partial^t f}{\partial^t x^k}}{\sqrt{\frac{\partial\, {}^t f}{\partial\, {}^t x^l} \, \frac{\partial\, {}^t f}{\partial\, {}^t x^m} \, {}^t g^{lm}}} \, . \tag{4.9f} $$

Using Eqs. (4.9e), (4.9f) and the first equality in Eq.(4.8), we obtain

$$ {}^{t}\dot{f} = \sqrt{\frac{\partial\, {}^t f}{\partial\, {}^t x^l} \, \frac{\partial\, {}^t f}{\partial\, {}^t x^m} \, {}^t g^{lm}} \, \left({}^t v_n - {}^t w_n \right) . \tag{4.9g} $$

If the particle instantaneously at P remains on the surface during the motion, from obvious geometrical considerations

$$ {}^{t}v_n = {}^{t}w_n \, , \tag{4.9h} $$

and using the above in Eq.(4.9g), we have for a *material surface*

$$ {}^{t}\dot{f} = 0 \, , \tag{4.10} $$

which demonstrates the Lagrange criterion.

Equation (4.9h) indicates that if we consider in the spatial configuration at time t *fictitious particles* (Truesdell & Toupin 1960) moving with a velocity field ${}^t\underline{\mathbf{w}}$, the volume $\Omega(t)$ and the surface $\sigma(t)$ can be considered material and we can write, using Eq.(4.6) for any Eulerian tensor field ${}^t\psi$,

$$ \frac{D_{{}^t\underline{\mathbf{w}}}}{Dt} \int_{\Omega(t)} {}^t\boldsymbol{\psi} \, {}^t\mathrm{d}V = \int_{\Omega(t)} \frac{\partial\, {}^t\psi}{\partial t} \, {}^t\mathrm{d}V + \int_{\sigma(t)} {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\mathbf{w}} \, {}^t\boldsymbol{\psi} \, {}^t\mathrm{d}S \, . \tag{4.11} $$

With $D_{{}^t\underline{\mathbf{w}}} (\cdot)/Dt$ we indicate that when taking the material time derivative, the velocity field ${}^t\underline{\mathbf{w}}$ is considered.

Equation (4.11) is the expression of the *generalized Reynolds' transport theorem.* An application of this theorem is presented in Example 4.5.

4.1.2 The transport theorem and discontinuity surfaces

In Fig. 4.2 we represent a body $\mathcal{B}$ in its spatial configuration corresponding to time t. We assume that the Eulerian tensor field ${}^t\boldsymbol{\psi}$ has a *jump discontinuity* across a surface ${}^tS_{12}$ inside the body and we let the material velocity field ${}^t\underline{\mathbf{v}}$ to be also discontinuous across ${}^tS_{12}$ (Truesdell & Toupin 1960). At any point on the *discontinuity surface* we define its normal (${}^t\underline{\mathbf{n}}_{12}$) and its displacement velocity (${}^t\underline{\mathbf{w}}$), not necessarily coincident with the material velocity of the particle instantaneously at that point (${}^t\underline{\mathbf{v}}$).

Considering the region on the negative side of ${}^t\underline{\mathbf{n}}_{12}$, we can define fictitious particles with the following velocity field:

- ${}^t\underline{\mathbf{v}}$ on ${}^tS^-$

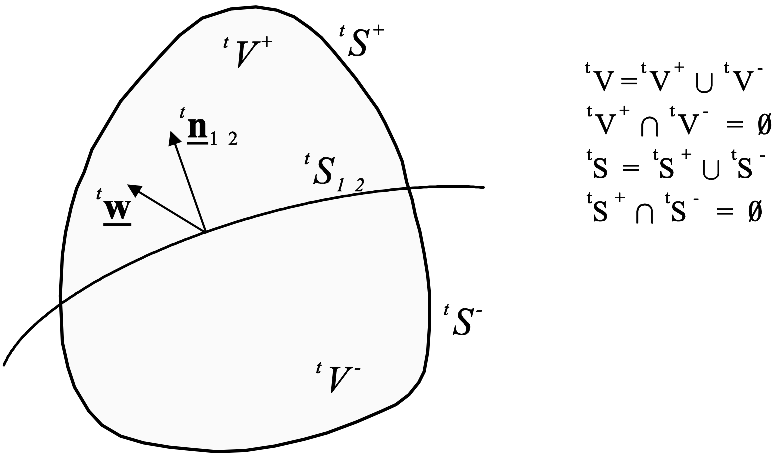

Fig. 4.2. Discontinuity surface

- $^t\underline{\mathbf{w}}$ on $^tS_{12}$

and using the generalized Reynolds' transport theorem we obtain,

$$\begin{aligned} \frac{D_{^t\underline{\mathbf{w}}}}{Dt}\int_{^tV^-} {}^t\boldsymbol{\psi}\ {}^t\mathrm{d}V \ &= \int_{^tV^-} \frac{\partial\ {}^t\boldsymbol{\psi}}{\partial t}\ {}^t\mathrm{d}V \ + \int_{^tS^-} {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi}\ {}^t\mathrm{d}S \\ &+ \int_{^tS_{12}} {}^t\underline{\mathbf{n}}_{12} \cdot {}^t\underline{\mathbf{w}}\ {}^t\boldsymbol{\psi}^-\ {}^t\mathrm{d}S\ . \end{aligned} \tag{4.12a}$$

Using the generalized Gauss' theorem (Eq.(4.5)),

$$\begin{aligned} \int_{^tV^-} \underline{\boldsymbol{\nabla}} \cdot ({}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi})\ {}^t\mathrm{d}V \ &= \int_{^tS^-} {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi}\ {}^t\mathrm{d}S \\ &+ \int_{^tS_{12}} {}^t\underline{\mathbf{n}}_{12} \cdot {}^t\underline{\mathbf{v}}^-\ {}^t\boldsymbol{\psi}^-\ {}^t\mathrm{d}S \end{aligned} \tag{4.12b}$$

hence,

$$\begin{aligned} \frac{D_{^t\underline{\mathbf{w}}}}{Dt}\int_{^tV^-} {}^t\boldsymbol{\psi}\ {}^t\mathrm{d}V \ &= \int_{^tV^-} \left[\frac{\partial^t\boldsymbol{\psi}}{\partial t} + \underline{\boldsymbol{\nabla}} \cdot ({}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi})\right] {}^t\mathrm{d}V \\ &- \int_{^tS_{12}} {}^t\boldsymbol{\psi}^-\ {}^t\underline{\mathbf{n}}_{12} \cdot ({}^t\underline{\mathbf{v}}^- - {}^t\underline{\mathbf{w}})\ {}^t\mathrm{d}S\ . \end{aligned} \tag{4.12c}$$

For the region on the positive side of $^t\underline{\mathbf{n}}_{12}$, in the same way, we get

$$\begin{aligned} \frac{D_{^t\underline{\mathbf{w}}}}{Dt}\int_{^tV^+} {}^t\boldsymbol{\psi}\ {}^t\mathrm{d}V \ &= \int_{^tV^+} \left[\frac{\partial^t\boldsymbol{\psi}}{\partial t} + \underline{\boldsymbol{\nabla}} \cdot ({}^t\underline{\mathbf{v}}\ {}^t\boldsymbol{\psi})\right] {}^t\mathrm{d}V \\ &+ \int_{^tS_{12}} {}^t\boldsymbol{\psi}^+\ {}^t\underline{\mathbf{n}}_{12} \cdot ({}^t\underline{\mathbf{v}}^+ - {}^t\underline{\mathbf{w}})\ {}^t\mathrm{d}S\ . \end{aligned} \tag{4.12d}$$

The velocity field of the fictitious particles is coincident with the velocity field of the actual particles everywhere except on ${}^tS_{12}$. Therefore:

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\;{}^t\mathrm{d}V \;=\; \frac{D_{{}^t\underline{\mathbf{w}}}}{Dt}\int_{{}^tV-} {}^t\boldsymbol{\psi}\;{}^t\mathrm{d}V \;+\; \frac{D_{{}^t\underline{\mathbf{w}}}}{Dt}\int_{{}^tV+} {}^t\boldsymbol{\psi}\;{}^t\mathrm{d}V\;. \qquad (4.12\mathrm{e})$$

From Eqs.(4.12c) to (4.12e) we get,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\;{}^t\mathrm{d}V \;=\int_{{}^tV}\left[\frac{\partial^t\boldsymbol{\psi}}{\partial t} + \underline{\boldsymbol{\nabla}}\cdot({}^t\underline{\mathbf{v}}\;{}^t\boldsymbol{\psi})\right]{}^t\mathrm{d}V$$
$$+\int_{{}^tS_{12}} [\![\,{}^t\boldsymbol{\psi}\;({}^tv_n \;-\; {}^tw_n)]\!]\;{}^t\mathrm{d}S \qquad (4.13)$$

where,

$[\![{}^t\boldsymbol{\psi}\;({}^tv_n \;-\; {}^tw_n)]\!] \;=\; {}^t\boldsymbol{\psi}^+\;({}^tv_n^+ \;-\; {}^tw_n) \;-\; {}^t\boldsymbol{\psi}^-\;({}^tv_n^- \;-\; {}^tw_n)\;,$
${}^tv_n \;=\; {}^t\underline{\mathbf{n}}_{12}\;\cdot\;{}^t\underline{\mathbf{v}}\;,$
${}^tw_n \;=\; {}^t\underline{\mathbf{n}}_{12}\;\cdot\;{}^t\underline{\mathbf{w}}\;.$

In order to obtain a *localized version* of Reynolds' transport theorem at the discontinuity surface we consider the arbitrary material volume enclosed by the dashed line in Fig. 4.3.

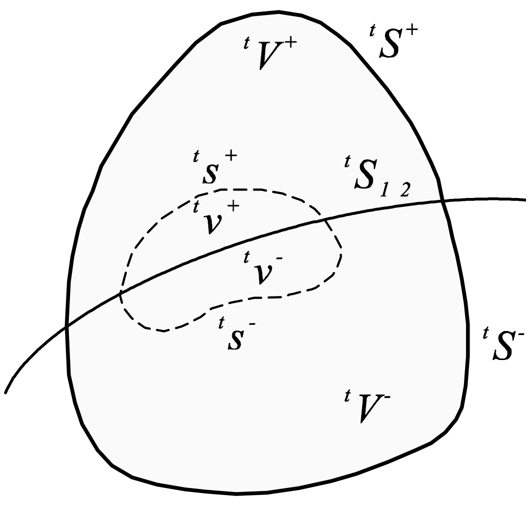

Fig. 4.3. Derivation of the jump discontinuity condition ${}^t\nu \;=\; {}^t\nu^+\;\cup\;{}^t\nu^-$

For the enclosed material volume, using Eq.(4.13) and the generalized Gauss' theorem, we write:

$$\frac{D}{Dt}\int_{{}^tV} {}^t\boldsymbol{\psi}\;{}^t\mathrm{d}V \;=\int_{{}^tV}\frac{\partial^t\boldsymbol{\psi}}{\partial t}\;{}^t\mathrm{d}V + \int_{{}^tS-} {}^t\underline{\mathbf{n}}\;\cdot\;{}^t\underline{\mathbf{v}}\;{}^t\boldsymbol{\psi}\;{}^t\mathrm{d}S \qquad (4.14)$$
$$+\int_{{}^tS+} {}^t\underline{\mathbf{n}}\;\cdot\;{}^t\underline{\mathbf{v}}\;{}^t\boldsymbol{\psi}\;{}^t\mathrm{d}S \;+\; \int_{{}^tS_{12}} [\![\,{}^t\boldsymbol{\psi}\;({}^tv_n \;-\; {}^tw_n)]\!]\;{}^t\mathrm{d}S,$$

when ${}^t\mathrm{d}S^+ \rightarrow^t S_{12}$ and ${}^t\mathrm{d}S^- \rightarrow^t S_{12}$; ${}^t\nu^+ \rightarrow 0$ and ${}^t\nu^- \rightarrow 0$ we get from Eq.(4.14):

$$\int_{{}^tS_{12}} \left([\![{}^t\boldsymbol{\psi}\ {}^tv_n]\!] \ + \ [\![{}^t\boldsymbol{\psi}\ ({}^tv_n \ - \ {}^tw_n)]\!] \right) \ {}^t\mathrm{d}S \ = \ 0 \ . \tag{4.15}$$

Therefore, in order for the above integral equation to be valid for any arbitrary part of the discontinuity surface, we must fulfill

$$[\![{}^t\boldsymbol{\psi}\ {}^tv_n]\!] \ + \ [\![{}^t\boldsymbol{\psi}\ ({}^tv_n \ - \ {}^tw_n)]\!] \ = \ 0 \tag{4.16a}$$

at every point on ${}^tS_{12}$.

Equation (4.16a) is known as the *jump discontinuity condition.*

If we call ${}^t\mathcal{U} = {}^tw_n - {}^tv_n$ the *discontinuity's propagation speed,* we can write

$$[\![{}^t\boldsymbol{\psi}\ {}^t\mathcal{U}]\!] \ = \ [\![{}^t\boldsymbol{\psi}\ {}^tv_n]\!] \ . \tag{4.16b}$$

The above equation is known as *Kotchine's theorem* (Truesdell & Toupin 1960).

4.2 Mass-conservation principle

In Sect. 2.2, Eq.(2.6) introduced the concept of *mass* of a continuum body $\mathcal{B}$.

In the study of continuum media, under the assumptions of Newtonian mechanics, it is postulated that *the mass of a continuum is conserved.* Hence,

$$\frac{D}{Dt} \int_{{}^tV} {}^t\rho \ {}^t\mathrm{d}V \ = \ 0 \tag{4.17}$$

where ${}^t\rho \ = \ {}^t\rho \ ({}^tx^i, t)$.

4.2.1 Eulerian (spatial) formulation of the mass-conservation principle

Using in Eq.(4.17) the expression of Reynolds' transport theorem given in Eq.(4.4) we obtain:

$$\frac{D}{Dt} \int_{{}^tV} {}^t\rho \ {}^t\mathrm{d}V \ = \ \int_{{}^tV} \left[\frac{\partial {}^t\rho}{\partial t} \ + \ \underline{\boldsymbol{\nabla}} \ \cdot \ ({}^t\rho \ {}^t\underline{\mathbf{v}}) \right] \ {}^t\mathrm{d}V \ = \ 0 \ . \tag{4.18}$$

Since the above equation has to be fulfilled for any control volume inside the continuum, we can write for any point inside the spatial configuration:

$$\frac{\partial {}^t\rho}{\partial t} \ + \ \underline{\boldsymbol{\nabla}} \ \cdot \ ({}^t\rho \ {}^t\underline{\mathbf{v}}) \ = \ 0 \ . \tag{4.19}$$

The above partial differential equation is the *localized spatial form* of the mass-conservation principle in a *Eulerian formulation* and it is called the *continuity equation.*

Example 4.3. ◀◀◀◀◀

Using components in a general curvilinear spatial coordinate system, the continuity equation is written as

$$\frac{\partial^t\rho}{\partial t} \; + \; {}^t v^a \, \frac{\partial^t\rho}{\partial^t x^a} \; + \; {}^t\rho \; {}^t v^a|_a \; = \; 0 \; .$$

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Example 4.4. ◀◀◀◀◀

For an incompressible material $\frac{D\ {}^t\rho}{Dt} \; = \; 0$; hence the continuity equation is:

$$\underline{\boldsymbol{\nabla}} \; \cdot \; {}^t\underline{\mathbf{v}} \; = \; 0 \; ,$$

or in components,

$${}^t v^a|_a \; = \; 0 \; .$$

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Example 4.5. ◀◀◀◀◀

Let a fluid of density ${}^t\rho \; = \; {}^t\rho \; ({}^t x^i, \; t)$ have a velocity field ${}^t\underline{\mathbf{v}}$.
Let us consider in the spatial configuration a volume $\Omega(t)$ bounded by a surface $\sigma(t)$ that moves with an arbitrary velocity field ${}^t\underline{\mathbf{w}}$.
Following (Thorpe 1962), we first calculate the fluid mass instantaneously inside the volume $\Omega(t)$:

$$M \; = \; \int_{\Omega(t)} {}^t\rho \; {}^t\mathrm{d}V$$

and using the expression of the generalized Reynolds' transport theorem in Eq.(4.11), we get

$$\frac{\mathrm{d}M}{\mathrm{d}t} \; = \; \frac{D_{{}^t\underline{\mathbf{w}}}}{Dt} \int_{\Omega(t)} {}^t\rho \; {}^t\mathrm{d}V \; = \; \int_{\Omega(t)} \frac{\partial^t\rho}{\partial t} \; {}^t\mathrm{d}V \; + \; \int_{\sigma(t)} {}^t\underline{\mathbf{n}} \; \cdot \; {}^t\underline{\mathbf{w}} \; {}^t\rho \; {}^t\mathrm{d}S$$

where ${}^t\underline{\mathbf{n}}$ is the external normal of the surface $\sigma(t)$.
Using Eq.(4.5) (generalized Gauss' theorem), we get

$$\int_{\Omega(t)} \underline{\boldsymbol{\nabla}} \; \cdot \; \left({}^t\rho \; {}^t\underline{\mathbf{v}}\right) \; {}^t\mathrm{d}V \; = \; \int_{\sigma(t)} {}^t\underline{\mathbf{n}} \; \cdot \; ({}^t\rho \; {}^t\underline{\mathbf{v}}) \; {}^t\mathrm{d}S$$

and subtracting the above equation from the previous one,

$$\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}t} \; = \; & \int_{\Omega(t)} \left[\frac{\partial^t\rho}{\partial t} \; + \; \underline{\boldsymbol{\nabla}} \; \cdot \; ({}^t\rho \; {}^t\underline{\mathbf{v}})\right] \; {}^t\mathrm{d}V \\ & + \; \int_{\sigma(t)} {}^t\rho \; {}^t\underline{\mathbf{n}} \; \cdot \; ({}^t\underline{\mathbf{w}} - {}^t\,\underline{\mathbf{v}}) \; {}^t\mathrm{d}S \quad . \end{aligned}$$

Using Eq.(4.19), we see that the first integral on the r.h.s. is zero; hence,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{\sigma(t)} {}^t\rho \; {}^t\underline{\mathbf{n}} \cdot ({}^t\underline{\mathbf{w}} - {}^t\,\underline{\mathbf{v}}) \; {}^t\mathrm{d}S \; .$$

The above equation is an integral equation of continuity for a control volume in motion in the fluid velocity field (Thorpe 1962). ——————◀◀◀◀◀

4.2.2 Lagrangian (material) formulation of the mass conservation principle

Equation (4.17) implies that,

$$\int_{{}^\circ V} {}^\circ\rho \; ({}^\circ x^A) \; {}^\circ\mathrm{d}V = \int_{{}^t V} {}^t\rho \; ({}^t x^a, t) \; {}^t\mathrm{d}V \tag{4.20a}$$

where $({}^\circ\rho,\ {}^\circ V)$ correspond to the reference configuration and $({}^t\rho,\ {}^t V)$ to the spatial configuration. Using Eq.(2.31) in the r.h.s. of Eq.(4.20a) and changing variables in the expression of ${}^t\rho$ we obtain,

$$\int_{{}^\circ V} {}^\circ\rho \; ({}^\circ x^A) \; {}^\circ\mathrm{d}V = \int_{{}^\circ V} {}^t\rho \; ({}^\circ x^A, t) \; {}^t J \; {}^\circ\mathrm{d}V \; , \tag{4.20b}$$

hence,

$$\int_{{}^\circ V} ({}^\circ\rho \; - \; {}^t\rho \; {}^t J) \; {}^\circ\mathrm{d}V = 0 \; . \tag{4.20c}$$

Since the above equation has to be fulfilled for any control volume that we define inside the continuum, we can write for any point inside the reference configuration:

$${}^\circ\rho = {}^t\rho \; {}^t J \; , \tag{4.20d}$$

and therefore,

$$\frac{D}{Dt} \, ({}^t\rho \; {}^t J) = 0 \; . \tag{4.20e}$$

The above equation is the *localized material form* of the *continuity equation*.

4.3 Balance of momentum principle (Equilibrium)

The principle of balance of momentum is the expression of Newton's Second Law for continuum bodies. Quoting (Malvern 1969):

"*The momentum principle for a collection of particles states that the time rate of change of the total momentum of a* ***given set of particles*** *equals the vector sum of all the* ***external*** *forces acting on the particles of the set, provided Newton's Third Law of action and reaction governs the* ***internal*** *forces. The continuum form of this principle is a basic postulate of continuum mechanics*".

4.3.1 Eulerian (spatial) formulation of the balance of momentum principle

For a body $\mathcal{B}$ in the t-configuration we define its momentum as,

$$\int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{v}} \; {}^t\mathrm{d}V \tag{4.21a}$$

the resultant of the external forces acting on the elements of mass inside the body are, from Eq.(3.2):

$$\int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{b}} \; {}^t\mathrm{d}V \; , \tag{4.21b}$$

and the resultant of the external forces acting on the elements of the body's surface are, from Eq.(3.4):

$$\int_{{}^tS} {}^t\underline{\mathbf{t}} \; {}^t\mathrm{d}S \; . \tag{4.21c}$$

Using Eqs.(4.21a-4.21c), we can state Newton's Second Law for the body B as,

$$\frac{D}{Dt} \int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{v}} \; {}^t\mathrm{d}V = \int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{b}} \; {}^t\mathrm{d}V + \int_{{}^tS} {}^t\underline{\mathbf{t}} \; {}^t\mathrm{d}S \; . \tag{4.22}$$

Using the condition of equivalence between external forces and Cauchy stresses inside a continuum, defined in Eq.(3.7), we get:

$$\frac{D}{Dt} \int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{v}} \; {}^t\mathrm{d}V = \int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{b}} \; {}^t\mathrm{d}V + \int_{{}^tS} {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} \; {}^t\mathrm{d}S \; . \tag{4.23}$$

Using in the above the expression of Reynolds' transport theorem given in Eq.(4.4), we get

$$\int_{{}^tV} \left[\frac{\partial({}^t\rho \; {}^t\underline{\mathbf{v}})}{\partial t} + \underline{\boldsymbol{\nabla}} \cdot ({}^t\rho \; {}^t\underline{\mathbf{v}} \; {}^t\underline{\mathbf{v}}) \right] {}^t\mathrm{d}V = \int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{b}} \; {}^t\mathrm{d}V + \int_{{}^tS} {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} \; {}^t\mathrm{d}S \; . \tag{4.24}$$

From Example 4.2, we obtain

$$\underline{\boldsymbol{\nabla}} \cdot ({}^t\rho \; {}^t\underline{\mathbf{v}} \; {}^t\underline{\mathbf{v}}) = {}^t\underline{\mathbf{v}} \cdot \left[\underline{\boldsymbol{\nabla}} \, ({}^t\rho \; {}^t\underline{\mathbf{v}}) \right] + {}^t\rho \; {}^t\underline{\mathbf{v}} \, (\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}) \tag{4.25a}$$

also, from Eq.(2.20b), we get

$$\frac{D \, ({}^t\rho \; {}^t\underline{\mathbf{v}})}{Dt} = \frac{\partial \, ({}^t\rho \; {}^t\underline{\mathbf{v}})}{\partial t} + {}^t\underline{\mathbf{v}} \cdot \left[\underline{\boldsymbol{\nabla}} \, ({}^t\rho \; {}^t\underline{\mathbf{v}}) \right] \; , \tag{4.25b}$$

and, from Eq.(4.5) (Generalized Gauss' Theorem), we get

$$\int_{{}^tS} {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}}\, {}^t\mathrm{d}S = \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot ({}^t\underline{\underline{\boldsymbol{\sigma}}})\, {}^t\mathrm{d}V \quad . \tag{4.25c}$$

Using Eqs.(4.25a-4.25c) in Eq.(4.24) we arrive at the integral form of the Eulerian formulation of the balance of momentum principle:

$$\int_{{}^tV} \left[\frac{D}{Dt}\left({}^t\rho\, {}^t\underline{\mathbf{v}}\right) + {}^t\rho\, {}^t\underline{\mathbf{v}}\left(\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}\right)\right] {}^t\mathrm{d}V = \int_{{}^tV} \left[{}^t\rho\, {}^t\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}}\right] {}^t\mathrm{d}V \;. \tag{4.26}$$

Since the above equation has to be fulfilled for any control volume that we define inside the continuum, we can write for any point inside the spatial configuration:

$$\frac{D}{Dt}\left({}^t\rho\, {}^t\underline{\mathbf{v}}\right) + {}^t\rho\, {}^t\underline{\mathbf{v}}\left(\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}\right) = {}^t\rho\, {}^t\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} \tag{4.27a}$$

and using in the above the continuity equation, we have

$${}^t\rho\, \frac{D\, {}^t\underline{\mathbf{v}}}{Dt} = {}^t\rho\, {}^t\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} \;. \tag{4.27b}$$

The above equation is the *localized form* of the balance of momentum principle in an *Eulerian formulation* and it is known as the *equilibrium equation.*

Example 4.6. ◀◀◀◀◀

Using Eq.(A.62b), in the general Eulerian curvilinear system $\{{}^tx^a\}$, we get,

$$\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^t\sigma^{ab}|_a\, {}^t\underline{\mathbf{g}}_b$$

hence, using Eq.(A.55b),

$$\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} = \left[\frac{\partial^t\sigma^{ab}}{\partial^t x^a} + {}^t\sigma^{sb}\, {}^t\Gamma^a_{sa} + {}^t\sigma^{as}\, {}^t\Gamma^b_{sa}\right] {}^t\underline{\mathbf{g}}_b \;.$$

From the result in Example A.10, we can easily get

$${}^t\Gamma^m_{il} = \frac{1}{2}\, {}^tg^{mj} \left[\frac{\partial\, {}^tg_{ij}}{\partial\, {}^tx^l} + \frac{\partial\, {}^tg_{jl}}{\partial\, {}^tx^i} - \frac{\partial\, {}^tg_{li}}{\partial\, {}^tx^j}\right] \;.$$

Therefore,

$$\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} = \Bigg[\frac{\partial^t\sigma^{ab}}{\partial^t x^a} + \frac{1}{2}\left({}^t\sigma^{sb}\, {}^tg^{aj} + {}^t\sigma^{as}\, {}^tg^{bj}\right) \left(\frac{\partial\, {}^tg_{sj}}{\partial\, {}^tx^a} + \frac{\partial\, {}^tg_{ja}}{\partial\, {}^tx^s} - \frac{\partial\, {}^tg_{as}}{\partial\, {}^tx^j}\right)\Bigg] {}^t\underline{\mathbf{g}}_b \;.$$

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Example 4.7. ◀◀◀◀◀

A *perfect fluid* is defined as a continuum in which, at every point, and for any surface,

$$ {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^t\gamma\ {}^t\underline{\mathbf{n}} \ , $$

where ${}^t\gamma$ is a scalar (*no shear stresses*).

Since ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ is a symmetric second order tensor (to be shown in Sect. 4.4), its eigenvalues are real and its eigenvectors are orthogonal (Appendix, A.4.1). Referring the problem to the Cartesian system defined by the normalized eigenvectors, ${}^t\underline{\hat{\mathbf{e}}}_\alpha$ we can write,

$$ \begin{aligned} {}^t\underline{\mathbf{n}} &= {}^t\hat{n}_\alpha\ {}^t\underline{\hat{\mathbf{e}}}_\alpha \\ {}^t\underline{\underline{\boldsymbol{\sigma}}} &= \sum_{\beta=1}^{3} {}^t\hat{\sigma}_{\beta\beta}\ {}^t\underline{\hat{\mathbf{e}}}_\beta\ {}^t\underline{\hat{\mathbf{e}}}_\beta \ . \end{aligned} $$

Then, for the perfect fluid,

$$ {}^t\hat{n}_\beta\ {}^t\hat{\sigma}_{\beta\beta} = {}^t\gamma\ {}^t\hat{n}_\beta \qquad (\beta = 1,2,3)(\textit{no addition on } \beta) \ . $$

The above set of equations is fulfilled only if the three eigenvalues ${}^t\hat{\sigma}_{\beta\beta}$ are equal (*hydrostatic stress tensor*). Hence,

$$ {}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^tp\ {}^t\underline{\hat{\mathbf{e}}}_\beta\ {}^t\underline{\hat{\mathbf{e}}}_\beta \ . $$

It is easy to show that as the three eigenvalues of ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ are equal, the above equation is valid in any Cartesian system; hence we can write,

$$ {}^t\sigma_{ij} = {}^tp\ \delta_{ij} \ , $$

where ${}^tp$ is the pressure. Generalizing the above to any arbitrary coordinate system

$$ {}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^tp\ {}^t\underline{\underline{\mathbf{g}}} \ . $$

Using Eq. (A.62b) and the result in Example A.11, the equilibrium equation, Eq. (4.27b), can be written as,

$$ {}^t\rho\ \frac{D\,{}^t\underline{\mathbf{v}}}{Dt} = {}^t\rho\ {}^t\underline{\mathbf{b}} + \frac{\partial p}{\partial\ {}^tx^i}\ {}^tg^{ij}\ {}^t\underline{\mathbf{g}}_j \ . $$

From Eq. (A.57) we identify the last term on the r.h.s. of the above equation as $\underline{\boldsymbol{\nabla}}{}^tp$, hence we can write the *equilibrium equation for a perfect fluid* as,

$$ {}^t\rho\ \frac{D\ {}^t\underline{\mathbf{v}}}{Dt} = {}^t\rho\ {}^t\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}}{}^tp \ . $$

The above equation is known as the *Euler equation for perfect fluids*. Many authors get a minus sign for the second term on the r.h.s. because they define

$$ {}^t\sigma_{ij} = -\ {}^tp\ \delta_{ij} \ . $$

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Example 4.8. ◀◀◀◀◀◀

Following with the topic discussed in Example 4.5 we consider a fluid, moving with a velocity field ${}^t\underline{\mathbf{v}}$, and a moving control volume, moving with a velocity field ${}^t\underline{\mathbf{w}}$. In this example, following (Thorpe 1962), we are going to analyze the momentum balance inside the moving control volume. Using the generalized Reynolds' transport theorem (Eq.(4.11)) for the fluid momentum,

$$\begin{aligned}\frac{D_{{}^t\underline{\mathbf{w}}}}{Dt}\int_{\Omega(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, {}^t\mathrm{d}V \;=\; &\int_{\Omega(t)} \frac{\partial({}^t\rho\, {}^t\underline{\mathbf{v}})}{\partial t}\, {}^t\mathrm{d}V \\ &+ \int_{\sigma(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, ({}^t\underline{\mathbf{n}} \cdot {}^t\underline{\mathbf{w}})\, {}^t\mathrm{d}S\,.\end{aligned}$$

From the generalized Gauss' theorem (Eq.(4.5)),

$$\int_{\Omega(t)} \underline{\boldsymbol{\nabla}} \cdot ({}^t\rho\, {}^t\underline{\mathbf{v}}\, {}^t\underline{\mathbf{v}})\, {}^t\mathrm{d}V \;=\; \int_{\sigma(t)} {}^t\underline{\mathbf{n}} \cdot ({}^t\rho\, {}^t\underline{\mathbf{v}}\, {}^t\underline{\mathbf{v}})\, {}^t\mathrm{d}S\,.$$

Subtracting the above equation from the previous one,

$$\begin{aligned}\frac{D_{{}^t\underline{\mathbf{w}}}}{Dt}\int_{\Omega(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, {}^t\mathrm{d}V \;=\; &\int_{\Omega(t)} \left[\frac{\partial({}^t\rho {}^t\underline{\mathbf{v}})}{\partial t} + \underline{\boldsymbol{\nabla}} \cdot ({}^t\rho {}^t\underline{\mathbf{v}} {}^t\underline{\mathbf{v}})\right] {}^t\mathrm{d}V \\ &+ \int_{\sigma(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, \left[{}^t\underline{\mathbf{n}} \cdot ({}^t\underline{\mathbf{w}} - {}^t\underline{\mathbf{v}})\right]\, {}^t\mathrm{d}S\,.\end{aligned}$$

Using the result in Example 4.2 and Eq.(4.19) (continuity equation),

$$\begin{aligned}\frac{D_{{}^t\underline{\mathbf{w}}}}{Dt}\int_{\Omega(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, {}^t\mathrm{d}V \;=\; &\int_{\Omega(t)} {}^t\rho \left(\frac{\partial {}^t\underline{\mathbf{v}}}{\partial t} + {}^t\underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}}\, {}^t\underline{\mathbf{v}}\right) {}^t\mathrm{d}V \\ &+ \int_{\sigma(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, \left[{}^t\underline{\mathbf{n}} \cdot ({}^t\underline{\mathbf{w}} - {}^t\underline{\mathbf{v}})\right]\, {}^t\mathrm{d}S\,.\end{aligned}$$

On the r.h.s. of the above equation, the term between the brackets in the first integral is the fluid particles material acceleration. We can state, using Newton's second law, that the external force instantaneously acting on the particles inside $\Omega(t)$ is,

$${}^t\underline{\mathbf{F}} \;=\; \int_{\Omega(t)} {}^t\rho\, {}^t\underline{\mathbf{a}}\, {}^t\mathrm{d}V\,.$$

Hence,

$${}^t\underline{\mathbf{F}} \;=\; \frac{D_{\underline{\mathbf{w}}}}{Dt}\int_{\Omega(t)} {}^t\rho\, {}^t\underline{\mathbf{v}}\, {}^t\mathrm{d}V + \int_{\sigma(t)} {}^t\rho^t\, \underline{\mathbf{v}}\, \left[{}^t\underline{\mathbf{n}} \cdot ({}^t\underline{\mathbf{v}} - {}^t\underline{\mathbf{w}})\right]\, {}^t\mathrm{d}S\,.$$

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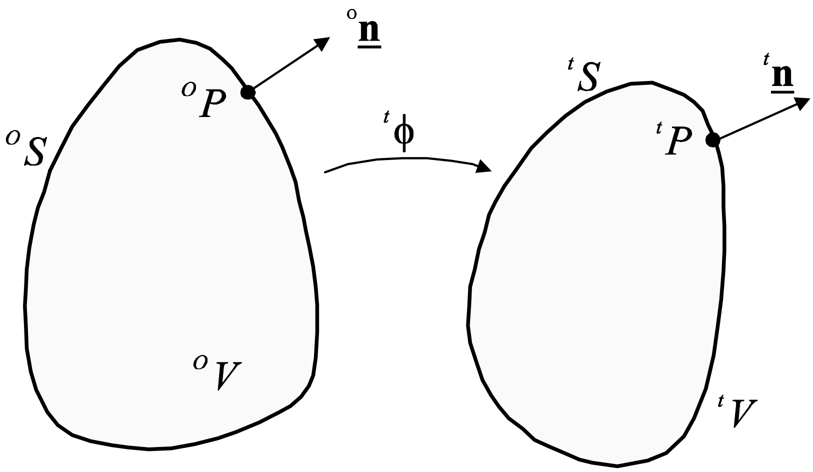
Example 4.9. ◀◀◀◀◀

Let us consider the body $\mathcal{B}$ and the particle P on its external surface. We define at P a convected coordinate system θ^i with covariant base vectors ${}^t\underline{\widetilde{\mathbf{g}}}_i$ in the spatial configuration and ${}^\circ\underline{\widetilde{\mathbf{g}}}_i$ in the material configuration. The convected system is defined so as to have ${}^t\underline{\widetilde{\mathbf{g}}}_1$ and ${}^t\underline{\widetilde{\mathbf{g}}}_2$ in the plane tangent to ${}^tS$ at ${}^tP$; and therefore ${}^\circ\underline{\widetilde{\mathbf{g}}}_1$ and ${}^\circ\underline{\widetilde{\mathbf{g}}}_2$ define the plane tangent to ${}^\circ S$ at ${}^\circ P$.



Material and spatial normal vectors (Nanson's formula)

The external unit normals at P are

$$ {}^t\underline{\mathbf{n}} = \frac{{}^t\underline{\widetilde{\mathbf{g}}}_1 \times {}^t\underline{\widetilde{\mathbf{g}}}_2}{|{}^t\underline{\widetilde{\mathbf{g}}}_1 \times {}^t\underline{\widetilde{\mathbf{g}}}_2|} , $$

and,

$$ {}^\circ\underline{\mathbf{n}} = \frac{{}^\circ\underline{\widetilde{\mathbf{g}}}_1 \times {}^\circ\underline{\widetilde{\mathbf{g}}}_2}{|{}^\circ\underline{\widetilde{\mathbf{g}}}_1 \times {}^\circ\underline{\widetilde{\mathbf{g}}}_2|} . $$

Also, the surface-area differentials are

$$ \begin{aligned} {}^t\mathrm{d}S\ {}^t\underline{\mathbf{n}} &= ({}^t\underline{\widetilde{\mathbf{g}}}_1 \times {}^t\underline{\widetilde{\mathbf{g}}}_2)\ \mathrm{d}\theta^1\ \mathrm{d}\theta^2 && (A) , \\ {}^\circ\mathrm{d}S\ {}^\circ\underline{\mathbf{n}} &= ({}^\circ\underline{\widetilde{\mathbf{g}}}_1 \times {}^\circ\underline{\widetilde{\mathbf{g}}}_2)\ \mathrm{d}\theta^1\ \mathrm{d}\theta^2 && (B) . \end{aligned} $$

If we define,

$$ \begin{aligned} {}^t\underline{\mathbf{t}}_1 &= \mathrm{d}\theta^1\ {}^t\underline{\widetilde{\mathbf{g}}}_1 , \\ {}^t\underline{\mathbf{t}}_2 &= \mathrm{d}\theta^2\ {}^t\underline{\widetilde{\mathbf{g}}}_2 , \\ {}^\circ\underline{\mathbf{t}}_1 &= \mathrm{d}\theta^1\ {}^\circ\underline{\widetilde{\mathbf{g}}}_1 , \\ {}^\circ\underline{\mathbf{t}}_2 &= \mathrm{d}\theta^2\ {}^\circ\underline{\widetilde{\mathbf{g}}}_2 , \end{aligned} $$

it is obvious from the results in Sect. 2.9.1 that

$$
\begin{aligned}
{}^{\circ}\underline{\mathbf{t}}_1 &= {}^{t}\underline{\mathbf{T}}_1^{\sharp} \\
{}^{\circ}\underline{\mathbf{t}}_2 &= {}^{t}\underline{\mathbf{T}}_2^{\sharp} \, .
\end{aligned}
$$

We can now define an arbitrary curvilinear system $\{{}^{t}x^{i}\}$ in the spatial configuration and another one $\{{}^{\circ}x^{I}\}$ in the material configuration, with covariant base vectors ${}^{t}\underline{\mathbf{g}}_i$ and ${}^{\circ}\underline{\mathbf{g}}_I$ respectively.

$$
\begin{aligned}
{}^{t}\underline{\mathbf{t}}_1 &= ({}^{t}t_1)^k \; {}^{t}\underline{\mathbf{g}}_k \\
{}^{t}\underline{\mathbf{t}}_2 &= ({}^{t}t_2)^k \; {}^{t}\underline{\mathbf{g}}_k \\
{}^{t}\underline{\mathbf{T}}_1^{\sharp} &= ({}^{t}t_1)^k \; ({}^{t}_{\circ}X^{-1})^{K}_{\;\;k} \; {}^{\circ}\underline{\mathbf{g}}_K \\
{}^{t}\underline{\mathbf{T}}_2^{\sharp} &= ({}^{t}t_2)^k \; ({}^{t}_{\circ}X^{-1})^{K}_{\;\;k} \; {}^{\circ}\underline{\mathbf{g}}_K \\
{}^{t}\underline{\mathbf{n}} &= {}^{t}n_i \; {}^{t}\underline{\mathbf{g}}^i \\
{}^{\circ}\underline{\mathbf{n}} &= {}^{\circ}n_I \; {}^{\circ}\underline{\mathbf{g}}^I
\end{aligned}
$$

where,

$$
{}^{t}_{\circ}X^{i}_{\;\;I} = \frac{\partial^{t}x^{i}}{\partial^{\circ}x^{I}} \, .
$$

We write Eqs.(A) and (B) using the above as,

$$
\begin{aligned}
{}^{t}\mathrm{d}S \; {}^{t}n_i &= {}^{t}\epsilon_{ijk} \; ({}^{t}t_1)^j \; ({}^{t}t_2)^k \\
{}^{\circ}\mathrm{d}S \; {}^{\circ}n_I &= {}^{\circ}\epsilon_{IJK} \; ({}^{t}t_1)^j \; ({}^{t}t_2)^k \; ({}^{t}_{\circ}X^{-1})^{J}_{\;\;j} \; ({}^{t}_{\circ}X^{-1})^{K}_{\;\;k} \, .
\end{aligned}
$$

Multiplying both sides of the above equation by $({}^{t}_{\circ}X^{-1})^{I}_{\;\;i}$, we get

$$
{}^{\circ}\mathrm{d}S \; {}^{\circ}n_I \; ({}^{t}_{\circ}X^{-1})^{I}_{\;\;i} = {}^{\circ}\epsilon_{IJK} \; ({}^{t}_{\circ}X^{-1})^{I}_{\;\;i} \; ({}^{t}_{\circ}X^{-1})^{J}_{\;\;j} \;\; ({}^{t}_{\circ}X^{-1})^{K}_{\;\;k} \;\; ({}^{t}t_1)^j \; ({}^{t}t_2)^k
$$

but, from Eqs.(A.37e) and (2.34g)

$$
{}^{\circ}\epsilon_{IJK} = e_{IJK} \sqrt{| \, {}^{\circ}g_{AB}|} \, .
$$

Hence, using Eq.(A.37c), we get

$$
{}^{\circ}\mathrm{d}S \; {}^{\circ}n_I \; ({}^{t}_{\circ}X^{-1})^{I}_{\;\;i} = \sqrt{| \, {}^{\circ}g_{AB}|} \; |{}^{t}_{\circ}X^{-1}| \; e_{ijk} \; ({}^{t}t_1)^j \; ({}^{t}t_2)^k
$$

and again using Eq.(A.37e), we get

$$
{}^{\circ}\mathrm{d}S \; {}^{\circ}n_I \; ({}^{t}_{\circ}X^{-1})^{I}_{\;\;i} = \sqrt{| \, {}^{\circ}g_{AB}|} \; |{}^{t}_{\circ}X^{-1}| \; \frac{{}^{t}\epsilon_{ijk}}{\sqrt{|{}^{t}g_{ab}|}} \; ({}^{t}t_1)^j \; ({}^{t}t_2)^k \, .
$$

Therefore,

$$ {}^{\circ}\mathrm{d}S\ {}^{\circ}n_I\ ({}^{t}_{\circ}X^{-1})^{I}_{\ i} \ = \ {}^{t}\mathrm{d}S\ {}^{t}n_i\ |{}^{t}_{\circ}X^{-1}|\ \sqrt{\frac{|\ {}^{\circ}g_{AB}|}{|{}^{t}g_{ab}|}} $$

and using Eq.(2.34i), we get

$$ {}^{t}\underline{\mathbf{n}}\ {}^{t}\mathrm{d}S \ = \ {}^{t}J\ {}^{\circ}\underline{\mathbf{n}}\ \cdot\ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1}\ {}^{\circ}\mathrm{d}S\ . $$

The above equation is called Nanson's formula (Bathe 1996). ◀◀◀◀◀

Example 4.10. ◀◀◀◀◀

For an Eulerian vector ${}^{t}\underline{\mathbf{a}}$ we define, using Eq.(2.76a),

$$ {}^{t}\underline{\mathbf{A}}^{\sharp} \ = \ \left[\left({}^{t}_{\circ}X^{-1}\right)^{B}_{\ b}\ {}^{t}a^{b}\right]\ {}^{\circ}\underline{\mathbf{g}}_{B}\ . $$

Using the generalized Gauss' theorem (Eq.(4.5)),

$$ \int_{{}^{t}V} \underline{\boldsymbol{\nabla}}\ \cdot\ {}^{t}\underline{\mathbf{a}}\ {}^{t}\mathrm{d}V \ = \ \int_{{}^{t}S} {}^{t}\underline{\mathbf{n}}\ \cdot\ {}^{t}\underline{\mathbf{a}}\ {}^{t}\mathrm{d}S\ . $$

In the r.h.s. integral we introduce Nanson's formula derived in Example 4.9; hence,

$$ \begin{aligned} \int_{{}^{t}V} \underline{\boldsymbol{\nabla}}\ \cdot\ {}^{t}\underline{\mathbf{a}}\ {}^{t}\mathrm{d}V \ &= \ \int_{{}^{\circ}S} {}^{t}J\ {}^{\circ}\underline{\mathbf{n}}\ \cdot\ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1}\ \cdot\ {}^{t}\underline{\mathbf{a}}\ {}^{\circ}\mathrm{d}S \\ &= \ \int_{{}^{\circ}S} {}^{t}J\ {}^{\circ}\underline{\mathbf{n}}\ \cdot\ \left[\left({}^{t}_{\circ}X^{-1}\right)^{B}_{\ b}\ {}^{t}a^{b}\right]\ {}^{\circ}\underline{\mathbf{g}}_{B}\ {}^{\circ}\mathrm{d}S \\ &= \ \int_{{}^{\circ}S} {}^{\circ}\underline{\mathbf{n}}\ \cdot\ \left({}^{t}J\ {}^{t}\underline{\mathbf{A}}^{\sharp}\right)\ {}^{\circ}\mathrm{d}S\ . \end{aligned} $$

Using again the generalized Gauss' theorem,

$$ \int_{{}^{t}V} \underline{\boldsymbol{\nabla}}\ \cdot\ {}^{t}\underline{\mathbf{a}}\ {}^{t}\mathrm{d}V \ = \ \int_{{}^{\circ}V} DIV\ \left({}^{t}J\ {}^{t}\underline{\mathbf{A}}^{\sharp}\right)\ {}^{\circ}\mathrm{d}V\ . $$

With the notation $DIV(\cdot)$ we indicate a divergence in the reference configuration. Using in the r.h.s. integral Eq.(2.31), we get

$$ \int_{{}^{\circ}V} {}^{t}J\ \left(\underline{\boldsymbol{\nabla}}\ \cdot\ {}^{t}\underline{\mathbf{a}}\right)\ {}^{\circ}\mathrm{d}V \ = \ \int_{{}^{\circ}V} DIV\ \left({}^{t}J\ {}^{t}\underline{\mathbf{A}}^{\sharp}\right)\ {}^{\circ}\mathrm{d}V \quad . $$

The localized form of the above equation is known as the *Piola Identity* (Marsden & Hughes 1983),

$$ {}^{t}J\ \left(\underline{\boldsymbol{\nabla}}\ \cdot\ {}^{t}\underline{\mathbf{a}}\right) \ = \ DIV\ \left({}^{t}J\ {}^{t}\underline{\mathbf{A}}^{\sharp}\right)\ . $$

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Example 4.11. ◀◀◀◀◀

We can write the Piola Identity, derived in the above example, as:

$$ {}^tJ\ {}^ta^b|_b \ = \ \left[{}^tJ\ ({}^t_\circ X^{-1})^B_{\ c}\ {}^ta^c\right]|_B\ . $$

After some algebra, we get

$$ {}^tJ\ {}^ta^b|_b \ = \ \left[{}^tJ\ ({}^t_\circ X^{-1})^B_{\ c}\right]|_B\ \ {}^ta^c \ + \ {}^tJ\ {}^ta^c|_c \quad . $$

Hence,

$$ \left[{}^tJ\ ({}^t_\circ X^{-1})^B_{\ c}\right]|_B \ = \ 0 $$

that is to say

$$ DIV\ \left({}^tJ\ {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1}\right) \ = \ \underline{\mathbf{0}}\ . $$

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4.3.2 Lagrangian (material) formulation of the balance of momentum principle

In the previous section we derived, in the spatial configuration, the integral and localized forms of the balance of momentum principle (equilibrium equations).

In this section we are going to derive, in the material configuration, the integral and localized forms of the equilibrium equations.

We are going to refer Eq.(4.23) to volumes and surfaces defined in the material configuration, using Eq.(4.20d) and Nanson's formula (Example 4.9).

$$ \frac{D}{Dt}\int_{{}^\circ V} {}^\circ\rho\ {}^t\underline{\mathbf{v}}\ {}^\circ\mathrm{d}V \ = \ \int_{{}^\circ V} {}^\circ\rho\ {}^t\underline{\mathbf{b}}\ {}^\circ\mathrm{d}V \ + \ \int_{{}^\circ S} \frac{{}^\circ\rho}{{}^t\rho}\ {}^\circ\underline{\mathbf{n}}\ \cdot\ {}^t_\circ\underline{\underline{\mathbf{X}}}^{-1}\ \cdot\ {}^t\underline{\underline{\boldsymbol{\sigma}}}\ {}^\circ\mathrm{d}S\ . \tag{4.28a} $$

In the above equation, all magnitudes are written as functions of $({}^\circ x^I,\ t)$.

Using the generalized Gauss' theorem together with the definition of the first Piola-Kirchhoff stress tensor, we obtain

$$ \int_{{}^\circ V} \frac{D}{Dt}\ ({}^\circ\rho\ {}^t\underline{\mathbf{v}})\ {}^\circ\mathrm{d}V \ = \ \int_{{}^\circ V} {}^\circ\rho\ {}^t\underline{\mathbf{b}}\ {}^\circ\mathrm{d}V \ + \ \int_{{}^\circ V} DIV\ ({}^t_\circ\underline{\underline{\mathbf{P}}})\ {}^\circ\mathrm{d}V\ . \tag{4.28b} $$

In the above (Malvern 1969),

$$ \begin{aligned} DIV\ ({}^t_\circ\underline{\underline{\mathbf{P}}}) \ &= \ {}^t_\circ P^{Aa}|_A\ {}^t\underline{\mathbf{g}}_a \\ &= \ \left[\frac{\partial\ {}^t_\circ P^{Aa}}{\partial{}^\circ x^A} \ + \ {}^t_\circ P^{Da}\ {}^\circ\Gamma^A_{DA} \ + \ {}^t_\circ P^{Ad}\ {}^t_\circ X^i_A\ {}^t\Gamma^a_{id}\right]\ {}^t\underline{\mathbf{g}}_a\ . \end{aligned} $$

Equation (4.28b) is an integral form of the equilibrium equations. It is important to note that although the integrals are calculated on volumes defined

in the reference configuration, the equilibrium is established in the spatial configuration.

The corresponding localized form is,

$$ {}^{\circ}\rho \; \frac{D^t\underline{\mathbf{v}}}{Dt} \;=\; {}^{\circ}\rho \; {}^t\underline{\mathbf{b}} \;+\; DIV\; ({}^t_{\circ}\underline{\underline{\mathbf{P}}}) \; . \tag{4.29}$$

Example 4.12. ◀◀◀◀◀

From Eqs.(4.27b) and (4.29) we get,

$$ \frac{{}^{\circ}\rho}{{}^t\rho} \; \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} \;=\; DIV\; ({}^t_{\circ}\underline{\underline{\mathbf{P}}}) \; . $$

The above equation is a particular application of the Piola Identity. ◀◀◀◀◀

In order to write the equilibrium equations in terms of fully material tensors we have to pull-back Eq.(4.29).

For the material velocity field:

$$ {}^t\underline{\mathbf{V}}^{\sharp} \;=\; \left[{}^t\phi^{*}\; ({}^tv^a)\right]^A \; {}^{\circ}\underline{\mathbf{g}}_A \; . \tag{4.30}$$

A "*physical interpretation*" of the pull-back of the material velocity contravariant components was presented in Example 2.12.

In the same way, for the material acceleration,

$$ {}^t\underline{\mathbf{a}} \;=\; \frac{D^t\underline{\mathbf{v}}}{Dt} \;=\; {}^ta^a \; {}^t\underline{\mathbf{g}}_a \tag{4.31}$$

and

$$ \left[{}^t\underline{\mathbf{A}}^{\sharp}\right]^A \;=\; \left[{}^t\phi^{*}\; ({}^ta^a)\right]^A \;=\; {}^t\tilde{a}^A \tag{4.32}$$

the ${}^t\tilde{a}^A$ are the components of the material acceleration vector in the convected system $\{{}^{\circ}x^A\}$ (*convected acceleration* (Simo & Marsden 1984)).

For the external loads per unit mass, we define

$$ {}^t\underline{\mathbf{B}}^{\sharp} \;=\; \left[{}^t\phi^{*}\; ({}^tb^a)\right]^A \; {}^{\circ}\underline{\mathbf{g}}_A \; . \tag{4.33}$$

Since $DIV\; ({}^t_{\circ}\underline{\underline{\mathbf{P}}})$ is an Eulerian vector,

$$ \left[{}^t\phi^{*}\;\left({}^t_{\circ}P^{Ia}|_I\right)\right]^A \;=\; ({}^t_{\circ}X^{-1})^A_{\;\;a}\; {}^t_{\circ}P^{Ia}|_I \; . \tag{4.34}$$

Therefore, the pull-back of Eq.(4.29) is

$$ {}^{\circ}\rho \; {}^t\underline{\mathbf{A}}^{\sharp} \;=\; {}^{\circ}\rho \; {}^t\underline{\mathbf{B}}^{\sharp} \;+\; {}^t_{\circ}\underline{\underline{\mathbf{X}}}^{-1} \cdot DIV\; ({}^t_{\circ}\underline{\underline{\mathbf{P}}}) \; . \tag{4.35}$$

In a Lagrangian formulation, the above equation is the localized form of the equilibrium equations.

4.4 Balance of moment of momentum principle (Equilibrium)

Quoting (Malvern 1969) again:

"*In a collection of particles whose interactions are equal, opposite, and collinear forces, the time rate of change of the total moment of momentum for a given collection of particles is equal to the vector sum of the moments of the external forces acting on the system. In the absence of distributed couples, we postulate the same principle for a continuum*".

The condition of no distributed couples (nonpolar media) was introduced in Sect. 3.2.

4.4.1 Eulerian (spatial) formulation of the balance of moment of momentum principle

For a body $\mathcal{B}$ in the t-configuration we define its *moment of momentum* with respect to a given point O as,

$$\int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}} \; {}^t\mathrm{d}V \; , \tag{4.36a}$$

where

$${}^t\underline{\mathbf{r}} = {}^t\underline{\mathbf{x}} - {}^t\underline{\mathbf{x}}_\circ \; ;$$

and ${}^t\underline{\mathbf{x}}$; ${}^t\underline{\mathbf{x}}_\circ$ are the position vectors of an arbitrary point P and of the point O, respectively.

The resultant moment with respect to O of the external forces acting on the elements of mass inside the body is,

$$\int_{{}^tV} {}^t\rho \; {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}} \; {}^t\mathrm{d}V \; , \tag{4.36b}$$

and the resultant moment with respect to O of the external forces acting on the elements of the body's surface is,

$$\int_{{}^tS} {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{t}} \; {}^t\mathrm{d}S \; . \tag{4.36c}$$

Using Eqs.(4.36a-4.36c) we can state the *balance of moment of momentum principle* for the continuum body $\mathcal{B}$:

$$\frac{D}{Dt} \int_{{}^tV} {}^t\rho \, {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}} \, {}^t\mathrm{d}V = \int_{{}^tV} {}^t\rho \, {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}} \, {}^t\mathrm{d}V + \int_{{}^tS} {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{t}} \, {}^t\mathrm{d}S \; . \tag{4.37a}$$

With the condition of equivalence between external forces and Cauchy stresses inside a continuum defined in Eq.(3.7), we get

$$\frac{D}{Dt}\int_{{}^tV} {}^t\rho\, {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\, {}^t\mathrm{d}V = \int_{{}^tV} {}^t\rho\, {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}}\, {}^t\mathrm{d}V + \int_{{}^tS} {}^t\underline{\mathbf{r}} \times ({}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}})\, {}^t\mathrm{d}S\,. \tag{4.37b}$$

Example 4.13. ◀◀◀◀◀

The last integral on the r.h.s. of Eq.(4.37b) can be written as,

$$\int_{{}^tS} {}^t\underline{\mathbf{r}} \times \left({}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}}\right) {}^t\mathrm{d}S = -\int_{{}^tS} \left({}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}}\right) \times {}^t\underline{\mathbf{r}}\, {}^t\mathrm{d}S$$
$$= -\int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\underline{\boldsymbol{\sigma}}} \times {}^t\underline{\mathbf{r}}\right) {}^t\mathrm{d}V$$

where we used Eq. (4.5) (Generalized Gauss' Theorem).
For any second-order tensor, ${}^t\underline{\underline{\mathbf{d}}}$, and for any vector, ${}^t\underline{\mathbf{c}}$, we can write

$${}^t\underline{\underline{\mathbf{d}}} \times {}^t\underline{\mathbf{c}} = {}^td^{pq}\, {}^tc^r\, {}^t\underline{\mathbf{g}}_p\, {}^t\underline{\mathbf{g}}_q \times {}^t\underline{\mathbf{g}}_r = {}^td^{pq}\, {}^tc^r\, {}^t\epsilon_{qro}\, {}^t\underline{\mathbf{g}}_p\, {}^t\underline{\mathbf{g}}^o$$
$$= -\, {}^td^{pq}\, {}^tc^r\, {}^t\epsilon_{rqo}\, {}^t\underline{\mathbf{g}}_p\, {}^t\underline{\mathbf{g}}^o\,.$$

Also,

$${}^t\underline{\mathbf{c}} \times {}^t\underline{\underline{\mathbf{d}}}^T = {}^tc^l \left({}^td^T\right)^{mn} {}^t\underline{\mathbf{g}}_l \times {}^t\underline{\mathbf{g}}_m\, {}^t\underline{\mathbf{g}}_n = {}^tc^l\, {}^td^{nm}\, {}^t\epsilon_{lmo}\, {}^t\underline{\mathbf{g}}^o\, {}^t\underline{\mathbf{g}}_n$$

and

$$\left({}^t\underline{\mathbf{c}} \times {}^t\underline{\underline{\mathbf{d}}}^T\right)^T = {}^tc^l\, {}^td^{nm}\, {}^t\epsilon_{lmo}\, {}^t\underline{\mathbf{g}}_n\, {}^t\underline{\mathbf{g}}^o\,.$$

Hence,

$${}^t\underline{\underline{\mathbf{d}}} \times {}^t\underline{\mathbf{c}} = -\left({}^t\underline{\mathbf{c}} \times {}^t\underline{\underline{\mathbf{d}}}^T\right)^T\,.$$

Therefore,

$$\int_{{}^tS} {}^t\underline{\mathbf{r}} \times \left({}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}}\right) {}^t\mathrm{d}S = \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T {}^t\mathrm{d}V\,.$$

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Using the above result,

$$\frac{D}{Dt}\int_{{}^tV} {}^t\rho\, {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\, {}^t\mathrm{d}V \tag{4.38a}$$
$$= \int_{{}^tV} {}^t\rho\, {}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}}\, {}^t\mathrm{d}V + \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T {}^t\mathrm{d}V\,.$$

Using in the above equation the expression of Reynolds' transport theorem given in Eq.(4.4), we get

$$\int_{{}^tV} \left[\frac{\partial}{\partial t}\left({}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right) + \underline{\boldsymbol{\nabla}} \cdot \left({}^t\rho\,{}^t\underline{\mathbf{v}}\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)\right] {}^t\mathrm{d}V$$

$$= \int_{{}^tV} {}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}}\,{}^t\mathrm{d}V + \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T {}^t\mathrm{d}V\,. \quad (4.38\mathrm{b})$$

With the result in Example 4.2, we get

$$\underline{\boldsymbol{\nabla}} \cdot \left({}^t\rho\,{}^t\underline{\mathbf{v}}\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right) = {}^t\underline{\mathbf{v}} \cdot \left[\underline{\boldsymbol{\nabla}}\left({}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)\right] + \left({}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)\left(\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}\right) \quad (4.39)$$

using Eqs. (4.39) and (2.20b) we can write Eq.(4.38b) as,

$$\int_{{}^tV} \left[\frac{D\left({}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)}{Dt} + {}^t\rho\left({}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)\left(\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}\right)\right] {}^t\mathrm{d}V$$

$$= \int_{{}^tV} {}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}}\,{}^t\mathrm{d}V + \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T {}^t\mathrm{d}V$$

and therefore,

$$\int_{{}^tV} \left\{\left({}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)\left[\frac{D\,{}^t\rho}{Dt} + {}^t\rho\left(\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}\right)\right] + {}^t\rho\frac{D}{Dt}\left({}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{v}}\right)\right\} {}^t\mathrm{d}V$$

$$= \int_{{}^tV} {}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}}\,{}^t\mathrm{d}V + \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T {}^t\mathrm{d}V\,. \quad (4.40\mathrm{a})$$

Equation (4.19), the Eulerian continuity equation, can be written as,

$$\frac{D\,{}^t\rho}{Dt} + {}^t\rho\left(\underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{v}}\right) = 0 \quad (4.40\mathrm{b})$$

hence, introducing the above into Eq.(4.40a), we obtain

$$\int_{{}^tV} {}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{a}}\,{}^t\mathrm{d}V = \int_{{}^tV} {}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}}\,{}^t\mathrm{d}V + \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T {}^t\mathrm{d}V\,. \quad (4.40\mathrm{c})$$

For the Eulerian formulation, the above is the integral expression of the balance of moment of momentum principle.

From it, we obtain the localized form:

$${}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{a}} = {}^t\rho\,{}^t\underline{\mathbf{r}} \times {}^t\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot \left({}^t\underline{\mathbf{r}} \times {}^t\underline{\underline{\boldsymbol{\sigma}}}^T\right)^T\,. \quad (4.41)$$

4.4.2 Symmetry of Eulerian and Lagrangian stress measures

From the Eulerian localized form of the balance of moment of momentum principle, we will first derive the *symmetry of the Cauchy stress tensor*.

Using an intermediate result that we got in Example 4.13, we can write the localized form of the balance of moment of momentum as,

$$ {}^{t}\underline{\mathbf{r}} \times {}^{t}\rho \frac{D\,{}^{t}\underline{\mathbf{v}}}{Dt} - {}^{t}\underline{\mathbf{r}} \times {}^{t}\rho\, {}^{t}\underline{\mathbf{b}} = -\underline{\boldsymbol{\nabla}} \cdot \left({}^{t}\underline{\underline{\boldsymbol{\sigma}}} \times {}^{t}\underline{\mathbf{r}}\right) . \tag{4.42} $$

Example 4.14. ◀◀◀◀◀

The divergence of the cross product between a second order tensor, ${}^{t}\underline{\underline{\boldsymbol{\sigma}}}$, and a vector, ${}^{t}\underline{\mathbf{r}}$, is

$$
\begin{aligned}
\underline{\boldsymbol{\nabla}} \cdot \left({}^{t}\underline{\underline{\boldsymbol{\sigma}}} \times {}^{t}\underline{\mathbf{r}}\right) &= \frac{\partial}{\partial\,{}^{t}x^{n}}\, {}^{t}\underline{\mathbf{g}}^{n} \cdot \left[{}^{t}\underline{\underline{\boldsymbol{\sigma}}} \times {}^{t}\underline{\mathbf{r}}\right] \\
&= \left(\underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right) \times {}^{t}\underline{\mathbf{r}} + {}^{t}\underline{\mathbf{g}}^{n} \cdot \left[{}^{t}\underline{\underline{\boldsymbol{\sigma}}} \times {}^{t}\underline{\mathbf{g}}_{n}\right] \\
&= -\, {}^{t}\underline{\mathbf{r}} \times \left(\underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right) + {}^{t}\underline{\mathbf{g}}^{n} \cdot \left[{}^{t}\sigma^{ab}\, {}^{t}\underline{\mathbf{g}}_{a}\, {}^{t}\underline{\mathbf{g}}_{b} \times {}^{t}\underline{\mathbf{g}}_{n}\right] \\
&= -\, {}^{t}\underline{\mathbf{r}} \times \left(\underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right) + {}^{t}\underline{\mathbf{g}}^{n} \cdot \left[{}^{t}\sigma^{ab}\, {}^{t}\underline{\mathbf{g}}_{a}\, {}^{t}\epsilon_{bno}\, {}^{t}\underline{\mathbf{g}}^{o}\right] \\
&= -\, {}^{t}\underline{\mathbf{r}} \times \left(\underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right) + {}^{t}\epsilon_{bno}\, {}^{t}\sigma^{nb}\, {}^{t}\underline{\mathbf{g}}^{o} .
\end{aligned}
$$

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Using the above result in Eq.(4.42), we get

$$ {}^{t}\underline{\mathbf{r}} \times \left[{}^{t}\rho \frac{D\,{}^{t}\underline{\mathbf{v}}}{Dt} - {}^{t}\rho\, {}^{t}\underline{\mathbf{b}} - \underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\right] = -\, {}^{t}\epsilon_{bno}\, {}^{t}\sigma^{nb}\, {}^{t}\underline{\mathbf{g}}^{o} . \tag{4.43a} $$

Using the Eulerian localized equilibrium equation (Eq.(4.27b)),

$$ {}^{t}\epsilon_{bno}\, {}^{t}\sigma^{nb}\, {}^{t}\underline{\mathbf{g}}^{o} = \underline{\mathbf{0}} . \tag{4.43b} $$

Introducing in the above Eq. (A.37e), we can write

$$ \left|\frac{\partial\,{}^{t}z^{i}}{\partial\,{}^{t}x^{j}}\right| {}^{t}e_{bno}\, {}^{t}\sigma^{nb} = 0 \;\; (o = 1, 2, 3) \tag{4.43c} $$

hence,

$$
\begin{aligned}
(o = 1) \quad {}^{t}\sigma^{23} - {}^{t}\sigma^{32} &= 0 , \\
(o = 2) \quad {}^{t}\sigma^{31} - {}^{t}\sigma^{13} &= 0 , \\
(o = 3) \quad {}^{t}\sigma^{12} - {}^{t}\sigma^{21} &= 0 .
\end{aligned} \tag{4.43d}
$$

Equations (4.43d) show that,

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \;=\; {}^{t}\underline{\underline{\boldsymbol{\sigma}}}^{T} \;, \tag{4.44} $$

that is to say, they show that the Cauchy stress tensor is symmetric.

The symmetry of the Cauchy stress tensor implies the symmetry of the Kirchhoff stress tensor, the second Piola-Kirchhoff stress tensor and the stress tensor defined in Eq.(3.28).

For the first Piola-Kirchhoff stress tensor, which is not symmetric, we can define the following symmetry condition (see Eq.(3.15d)):

$$ {}^{t}_{\circ}X^{i}{}_{I}\; {}^{t}_{\circ}P^{Ij} \;=\; {}^{t}_{\circ}X^{j}{}_{I}\; {}^{t}_{\circ}P^{Ii} \;. \tag{4.45} $$

4.5 Energy balance (First Law of Thermodynamics)

To study the conservation of energy in our framework of Newtonian continuum mechanics, we will add to our set of variables a new one: the *internal energy*.

Quoting (Marsden & Hughes 1983) we can state that the internal energy "*represents energy stored internally in the body, which is a macroscopic reflection of things like chemical binding energy, intermolecular energy and energy of molecular vibrations*".

Examples of internal energy are:

- The energy stored in a deformed spring (elastic energy).
- The energy stored in a heated body (thermal energy).
- The energy stored in a bottle of oil (chemical energy).

In the next chapter, when we study the different material constitutive relations we will present some phenomenological relations between different forms of internal energy and the continuum state variables (stress, strain, temperature, etc.).

In the following subsections, we will derive the Eulerian (spatial) and Lagrangian (material) formulations of the First Law of Thermodynamics (Malvern 1969).

4.5.1 Eulerian (spatial) formulation of the energy balance

For a body $\mathcal{B}$ in the t-configuration we define ${}^{t}\mathtt{u}$ as its *internal energy per unit mass*. Being ${}^{t}K$ the kinetic energy of $\mathcal{B}$ defined by Eq.(3.9g), the total energy ${}^{t}E$ in the considered body, at the instant t, is

$$ {}^{t}E \;=\; {}^{t}K \;+\; \int_{{}^{t}V} {}^{t}\rho\; {}^{t}\mathtt{u}\; {}^{t}\mathrm{d}V \;. \tag{4.46} $$

The external forces acting on the body provide a mechanical power input ${}^{t}P_{ext}$. Using Eq.(3.9j), we write

$$ {}^tP_{ext} \; = \; \frac{D^tK}{Dt} \; + \; \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} \; : \; {}^t\underline{\underline{\mathbf{d}}} \; {}^t\mathrm{d}V \; . \tag{4.47} $$

We call "*heat*" (${}^tQ$) the energy that flows due to a temperature gradient. We will consider two types of heat:

- An outflowing *heat flux* through the body external surface (${}^t\underline{\mathbf{q}}$: heat flux vector). Examples are radiative and convective heat exchanges between the body $\mathcal{B}$ and the external medium.
- An internal distributed heat source per unit mass (${}^tr$). Examples are chemical reactions, phase changes, etc.

The total heat input to $\mathcal{B}$ at the instant t is,

$$ {}^tQ_{input} \; = \; - \int_{{}^tS} {}^t\underline{\mathbf{q}} \; \cdot \; {}^t\underline{\mathbf{n}} \; {}^t\mathrm{d}S \; + \; \int_{{}^tV} {}^t\rho \; {}^tr \; {}^t\mathrm{d}V \; . \tag{4.48} $$

Using the First Law of Thermodynamics, we can write

$$ \frac{D^tE}{Dt} \; = \; {}^tP_{ext} \; + \; {}^tQ_{input} \tag{4.49a} $$

hence,

$$ \begin{aligned} \frac{D}{Dt} \int_{{}^tV} {}^t\rho \; {}^t\mathtt{u} \; {}^t\mathrm{d}V \; = \; & \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} \; : \; {}^t\underline{\underline{\mathbf{d}}} \; {}^t\mathrm{d}V \; - \; \int_{{}^tS} {}^t\underline{\mathbf{q}} \; \cdot \; {}^t\underline{\mathbf{n}} \; {}^t\mathrm{d}S \\ & + \int_{{}^tV} {}^t\rho \; {}^tr \; {}^t\mathrm{d}V \; . \end{aligned} \tag{4.49b} $$

Using Gauss' theorem (Eq.(4.5)), we get

$$ \begin{aligned} \frac{D}{Dt} \int_{{}^tV} {}^t\rho \; {}^t\mathtt{u} \; {}^t\mathrm{d}V \; = \; & \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} \; : \; {}^t\underline{\underline{\mathbf{d}}} \; {}^t\mathrm{d}V \; - \; \int_{{}^tV} \underline{\boldsymbol{\nabla}} \; \cdot \; {}^t\underline{\mathbf{q}} \; {}^t\mathrm{d}V \\ & + \int_{{}^tV} {}^t\rho \; {}^tr \; {}^t\mathrm{d}V \; . \end{aligned} \tag{4.49c} $$

Using in the above equation the expression of Reynolds' transport theorem given in Eq.(4.4), we get

$$ \begin{aligned} & \int_{{}^tV} \left[\frac{\partial \left({}^t\rho \; {}^t\mathtt{u} \right)}{\partial t} \; + \; \underline{\boldsymbol{\nabla}} \cdot \; \left({}^t\underline{\mathbf{v}} \; {}^t\rho \; {}^t\mathtt{u} \right) \right] \; {}^t\mathrm{d}V \qquad (4.49\mathrm{d}) \\ = & \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} \; : \; {}^t\underline{\underline{\mathbf{d}}} \; {}^t\mathrm{d}V \; - \; \int_{{}^tV} \underline{\boldsymbol{\nabla}} \; \cdot \; {}^t\underline{\mathbf{q}} \; {}^t\mathrm{d}V \; + \; \int_{{}^tV} {}^t\rho \; {}^tr \; {}^t\mathrm{d}V \; . \end{aligned} $$

Now, we introduce in the integral on the l.h.s. of the above equation the result in Example 4.2, the Eulerian continuity equation in Eq.(4.19) and the definition of material derivative in Eq.(4.49d); hence,

$$\int_{{}^tV} {}^t\rho \, \frac{D^t\mathrm{u}}{Dt} \, {}^t\mathrm{d}V = \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}} \, {}^t\mathrm{d}V - \int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{q}} \, {}^t\mathrm{d}V + \int_{{}^tV} {}^t\rho \, {}^tr \, {}^t\mathrm{d}V \; . \tag{4.49e}$$

The above equation is the integral form of the Eulerian formulation for the energy conservation principle.

Since the above equation has to be fulfilled for any control volume that we define inside the continuum, we obtain the *localized form* of the energy conservation principle in the *Eulerian formulation*,

$$ {}^t\rho \, \frac{D^t\mathrm{u}}{Dt} = {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}} - \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{q}} + {}^t\rho \, {}^tr \; . \tag{4.50}$$

Example 4.15. ◀◀◀◀◀

Following from the topic discussed in Examples 4.5 and 4.8, we consider a fluid moving with a velocity field ${}^t\underline{\mathbf{v}}$ and a moving control volume with a velocity field ${}^t\underline{\mathbf{w}}$. In this example, we are going to analyze the energy balance inside the moving control volume. Using the generalized Reynolds' transport theorem (Eq. (4.11)) for the fluid internal energy,we get

$$\frac{D_{{}^t\underline{\mathbf{w}}}}{Dt} \int_{\Omega(t)} {}^t\rho \, {}^t\mathrm{u} \, {}^t\mathrm{d}V = \int_{\Omega(t)} \frac{\partial \left({}^t\rho {}^t\mathrm{u}\right)}{\partial t} \, {}^t\mathrm{d}V + \int_{\sigma(t)} {}^t\rho \, {}^t\mathrm{u} \left({}^t\underline{\mathbf{n}} \cdot {}^t\underline{\mathbf{w}}\right) \, {}^t\mathrm{d}S \quad .$$

From the generalized Gauss' theorem (Eq.(4.5)),

$$\int_{\Omega(t)} \underline{\boldsymbol{\nabla}} \cdot \left({}^t\rho \, {}^t\mathrm{u} \, {}^t\underline{\mathbf{v}}\right) \, {}^t\mathrm{d}V = \int_{\sigma(t)} {}^t\underline{\mathbf{n}} \cdot \left({}^t\rho \, {}^t\mathrm{u} \, {}^t\underline{\mathbf{v}}\right) \, {}^t\mathrm{d}S \quad .$$

Subtracting the above equation from the previous one,

$$\frac{D_{{}^t\underline{\mathbf{w}}}}{Dt} \int_{\Omega(t)} {}^t\rho \, {}^t\mathrm{u} \, {}^t\mathrm{d}V = \int_{\Omega(t)} \left[\frac{\partial \left({}^t\rho \, {}^t\mathrm{u}\right)}{\partial t} + \underline{\boldsymbol{\nabla}} \cdot \left({}^t\rho \, {}^t\mathrm{u} \, {}^t\underline{\mathbf{v}}\right)\right] {}^t\mathrm{d}V + \int_{\sigma(t)} {}^t\rho \, {}^t\mathrm{u} \, {}^t\underline{\mathbf{n}} \cdot \left({}^t\underline{\mathbf{w}} - {}^t\underline{\mathbf{v}}\right) \, {}^t\mathrm{d}S \; .$$

Using the result in Example 4.2 and Eq.(4.19) (continuity equation),

$$\frac{D_{{}^t\underline{\mathbf{w}}}}{Dt} \int_{\Omega(t)} {}^t\rho \, {}^t\mathrm{u} \, {}^t\mathrm{d}V = \int_{\Omega(t)} {}^t\rho \left[\frac{\partial^t\mathrm{u}}{\partial t} + {}^t\underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}} \left({}^t\mathrm{u}\right)\right] {}^t\mathrm{d}V + \int_{\sigma(t)} {}^t\rho \, {}^t\mathrm{u} \, {}^t\underline{\mathbf{n}} \cdot \left({}^t\underline{\mathbf{w}} - {}^t\underline{\mathbf{v}}\right) \, {}^t\mathrm{d}S \; .$$

On the r.h.s. of the above equation, the term between brackets in the first

integral is the material derivative of the fluid internal energy. Hence, using Eq.(4.50),

$$\begin{aligned}&\int_{\Omega(t)} \left[{}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}} - \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{q}} + {}^t\rho\, {}^tr\right] {}^t\mathrm{d}V \\ &= \frac{D_{{}^t\underline{\mathbf{w}}}}{Dt} \int_{{}^t\Omega} {}^t\rho\, {}^t\mathtt{u}\, {}^t\mathrm{d}V + \int_{\sigma(t)} {}^t\rho\, {}^t\mathtt{u}\, {}^t\underline{\mathbf{n}} \cdot \left({}^t\underline{\mathbf{v}} - {}^t\underline{\mathbf{w}}\right) {}^t\mathrm{d}S\,.\end{aligned}$$

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4.5.2 Lagrangian (material) formulation of the energy balance

Let us call ${}^t\mathtt{U}$ the internal energy per unit mass in the t-configuration but referred to the reference configuration. Obviously,

$$ {}^t\mathtt{U}\,({}^\circ x^I, t) = {}^t\mathtt{u}\,({}^t x^i, t)\,. \tag{4.51a}$$

From Eqs.(3.10) and (3.20c),

$$\int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}}\, {}^t\mathrm{d}V = \int_{{}^\circ V} {}^t_\circ\underline{\underline{\mathbf{S}}} : {}^t_\circ\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}\, {}^\circ\mathrm{d}V \tag{4.51b}$$

and using Eq.(2.76a), we define

$$ {}^t\underline{\mathbf{Q}}^\sharp = ({}^tQ^\sharp)^A\, {}^\circ\underline{\mathbf{g}}_A = \left[({}^t_\circ X^{-1})^A_{\ a}\, {}^tq^a\right] {}^\circ\underline{\mathbf{g}}_A\,. \tag{4.51c}$$

For the heat sources per unit mass we can also define in the reference configuration,

$$ {}^tR\left({}^\circ x^I, t\right) = {}^tr\ \left({}^t x^i, t\right)\,. \tag{4.51d}$$

Using the Piola Identity (Example 4.10), we can write

$$\int_{{}^tV} \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\mathbf{q}}\, {}^t\mathrm{d}V = \int_{{}^\circ V} DIV\left({}^tJ\, {}^t\underline{\mathbf{Q}}^\sharp\right) {}^\circ\mathrm{d}V\,. \tag{4.51e}$$

Hence, introducing into Eq.(4.49e) the results in Eq. (4.51a-4.51e), we get the integral form of the Lagrangian formulation for the energy conservation principle,

$$\begin{aligned}\int_{{}^\circ V} {}^\circ\rho\, \frac{D\,{}^t\mathtt{U}}{Dt}\, {}^\circ\mathrm{d}V =& \int_{{}^\circ V} {}^t_\circ\underline{\underline{\mathbf{S}}} : {}^t_\circ\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}\, {}^\circ\mathrm{d}V - \int_{{}^\circ V} DIV\left({}^tJ\, {}^t\underline{\mathbf{Q}}^\sharp\right) {}^\circ\mathrm{d}V \\ &+ \int_{{}^\circ V} {}^\circ\rho\, {}^tR\, {}^\circ\mathrm{d}V\,. \end{aligned}\tag{4.52}$$

Since the above equation has to be fulfilled for any volume that we define in the reference configuration, we get the *localized form* of the energy conservation principle in the *Lagrangian formulation*,

$$ {}^{\circ}\rho \, \frac{D^t \mathtt{U}}{Dt} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{S}}} \;:\; {}^{t}_{\circ}\underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \;-\; DIV \left({}^{t}J \; {}^{t}\underline{\mathbf{Q}}^{\sharp} \right) \;+\; {}^{\circ}\rho \; {}^{t}R \,. \tag{4.53} $$

Example 4.16. ◀◀◀◀◀

Using Eq.(3.18), we can write an alternative localized form of the energy conservation principle,

$$ {}^{\circ}\rho \, \frac{D^t \mathtt{U}}{Dt} \;=\; {}^{t}_{\circ}\underline{\underline{\mathbf{P}}}^{T} \;:\; {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{X}}}} \;-\; DIV \left({}^{t}J \; {}^{t}\underline{\mathbf{Q}}^{\sharp} \right) \;+\; {}^{\circ}\rho \; {}^{t}R \quad . $$

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5

Constitutive relations

Following (Marsden & Hughes 1983) we can state that the *constitutive relations in a continuum* are the functional forms that adopt the stress tensor, the free energy and the heat flow as functions of the continuum deformation and temperature.

Constitutive relations can be formulated using two different methodologies:

(i) Studying the phenomena that takes place on the atomic scale (deformation of the atomic lattices, movement of dislocations, etc. (Dieter 1986)).
(ii) Using phenomenological mathematical models that can match laboratory observations at the macroscopic scale.

The *phenomenological constitutive relations* are generally used in continuum mechanics and we will concentrate on this approach in this chapter.

This chapter is intended as an introduction to a large number of different constitutive models; hence, the recommended literature has to be classified into different areas:

- For studying the fundamentals that have to be considered for formulating constitutive relations some reference books are: (Truesdell & Noll 1965, Malvern 1969, Marsden & Hughes 1983).
- For studying hyperelasticity: (Ogden 1984).
- For studying plasticity: (Hill 1950, Mendelson 1968, Johnson & Mellor 1973, Lubliner 1990, Simo & Hughes 1998, Kojić & Bathe 2005).
- For studying viscoplasticity: (Perzyna 1966, Kojić & Bathe 2005).
- For studying viscoelasticity: (Pipkin 1972).
- For studying damage mechanics: (Lamaitre & Chaboche 1990).

5.1 Fundamentals for formulating constitutive relations

In this section, we will discuss some principles that shall be considered when developing a constitutive relation for modeling the behavior of any material.

5.1.1 Principle of equipresence

This principle has been proposed in (Truesdell & Toupin 1960) and very generally states that any independent variable that is included in the formulation of any constitutive relation for a given material has to be included in the formulation of all the other constitutive relations that are developed for the same material *unless* it is shown that its inclusion is either not necessary or violates some physical law.

Often, the constitutive models used by scientists and engineers are derived considering a very simplified material behavior and some variables are not included in the derived relation; e.g. usually when stating the relation between heat flux and temperature the continuum deformations are not considered (Fourier's Law). The principle of equipresence requires that these simplifications should not be made "by default" and that each of them should be specifically analyzed. These analyses will also be helpful for evaluating and understanding the limitations of the obtained results.

5.1.2 Principle of material-frame indifference

This principle states that the continuum constitutive relations shall be formulated using *objective physical laws* (see Section 2.12).

When using Cartesian coordinate systems in the spatial and material configurations, only classical objectivity is required. In more general cases, covariant formulations should be used.

Quoting (Ogden 1984), we can describe classical material objectivity as:

"*An important assumption in continuum mechanics is that two observers in relative motion make equivalent (mathematical and physical) deductions about the macroscopic properties of a material under test. In other words, material properties are unaffected by a superposed rigid motion, and the relation between the stress and the motion has the same form for all observers*".

5.1.3 Application to the case of a continuum theory restricted to mechanical variables

This theory considers some measure of the continuum deformations as the only independent variable and some measure of the continuum stresses as the only dependent variable.

Since many of the physical problems usually analyzed by scientists and engineers correspond to this category, it is an important case to be considered.

In (Truesdell & Noll 1965) the following principles are proposed for the specific case of a continuum theory restricted to mechanical variables:

- Stresses are deterministic functions of the continuum deformation history.
- *Local action*: the stresses acting on a material particle (material point) are only a function of the strains at the same material particle and not of the strains at neighboring particles. It is important to note that nowadays nonlocal continuum theories are used for very specific problems (Pijaudier-Cabot, Bažant & Tabbara 1988). Hence, we consider the principle of local action as a convenient hypothesis that, although it does not represent a physical law, provides a simplification to the constitutive relations that agrees with physical observations for many materials.

Example 5.1. ◀◀◀◀◀

Let us assume that for a problem in which only mechanical variables are considered, we formulate the following constitutive relation:

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t}\underline{\underline{\mathbf{m}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}\right) $$

where ${}^{t}\underline{\underline{\mathbf{m}}}$ is a tensorial function that maps the space of invertible two-point tensors into the space of symmetric Eulerian tensors (Ogden 1984). It is important to realize that ${}^{t}\underline{\underline{\mathbf{m}}}$ depends on the selected reference configuration.
Since we have restricted ourselves to study a purely mechanical problem we can accept that the proposed constitutive relation fulfills the principle of equipresence.
To study the objectivity of the formulation (classical objectivity) we consider in the spatial configuration two Cartesian coordinate systems: a fixed one $\{ {}^{t}z_{\alpha}\}$ and a moving one $\{ {}^{t}z^{*}_{\alpha}\}$.
Since the Cauchy stress is an objective spatial tensor, from Eq. (2.101c),

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}^{*} \cdot \underline{\underline{\mathbf{Q}}}^{T}(t) $$

where $\underline{\underline{\mathbf{Q}}}(t)$ is an orthogonal tensor.
And since the deformation gradient tensor is a two-point objective tensor, from Eq. (2.102c):

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{*} \; . $$

Since the above relation is valid for any orthogonal $\underline{\underline{\mathbf{Q}}}$, in particular it is also valid for ${}^{t}_{\circ}\underline{\underline{\mathbf{R}}}$ (from the polar decomposition ${}^{t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$).
The observer in the moving frame writes, due to material objectivity,

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}}^{*} = {}^{t}\underline{\underline{\mathbf{m}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{*}\right) $$

and using variables measured in the stationary frame,

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}}^{*} = {}^{t}\underline{\underline{\mathbf{m}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \right) = {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \; . $$

Hence,

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^{t}\underline{\underline{\mathbf{m}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{U}}}\right) \cdot \underline{\underline{\mathbf{Q}}}^{T}(t) \ . $$

The principle of material-frame indifference imposes ${}^{t}\underline{\underline{\mathbf{m}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{U}}}\right)$ instead of ${}^{t}\underline{\underline{\mathbf{m}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{X}}}\right)$.
The requirements of determinism and local action are obviously fulfilled. ◀◀◀◀◀

Example 5.2. ______________________________ ◀◀◀◀◀
As an alternative formulation to the one presented in Example 5.1, we assume that for a problem in which only mechanical variables are considered

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{T} \ = \ {}_{\circ}^{t}\underline{\underline{\mathbf{N}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{X}}}\right) $$

where ${}_{\circ}^{t}\underline{\underline{\mathbf{N}}}$ is a tensorial function that maps the space of invertible two-point tensors into a space of general two-point tensors. The relation between ${}^{t}\underline{\underline{\mathbf{m}}}$ (in Example 5.1) and ${}_{\circ}^{t}\underline{\underline{\mathbf{N}}}$ can be derived using Eq. (3.15d).
For the fulfillment of the principle of equipresence, we make the same comment as in the above example.
Again, to study the objectivity of the formulation, we consider in the spatial configuration two Cartesian systems: a fixed one $\{{}^{t}z_{\alpha}\}$ and a moving one $\{{}^{t}z_{\alpha}^{*}\}$.
Since the first Piola-Kirchhoff stress tensor is an objective two-points tensor, from Eq. (2.102c):

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{T} \ = \ \underline{\underline{\mathbf{Q}}}\,(t) \ \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{*T} \ , $$

where $\underline{\underline{\mathbf{Q}}}\,(t)$ is an orthogonal tensor.
In the same way

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} \ = \ \underline{\underline{\mathbf{Q}}}\,(t) \ \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{*} \ . $$

The observer in the moving frame writes, due to material objectivity,

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{*T} \ = \ {}_{\circ}^{t}\underline{\underline{\mathbf{N}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{*}\right) \ , $$

and particularizing for $\underline{\underline{\mathbf{Q}}}\,(t) \ = \ {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}$ (from the polar decomposition ${}_{\circ}^{t}\underline{\underline{\mathbf{X}}} \ = \ {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}$),

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{*T} \ = \ {}_{\circ}^{t}\underline{\underline{\mathbf{N}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{U}}}\right) \ = \ {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{T} \ . $$

Hence,

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{P}}}^{T} \ = \ \underline{\underline{\mathbf{Q}}}(t) \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{N}}}\left({}_{\circ}^{t}\underline{\underline{\mathbf{U}}}\right) \ . $$

The principle of material-frame indifference imposes the above form for the constitutive relation. ______________________________ ◀◀◀◀◀

Example 5.3. ◀◀◀◀◀

Another alternative formulation for a pure mechanical problem is,

$$ {}^t_\circ\underline{\underline{\mathbf{S}}} = {}^t_\circ\underline{\underline{\mathbf{M}}}\left({}^t_\circ\underline{\underline{\mathbf{X}}}\right) $$

where ${}^t_\circ\underline{\underline{\mathbf{M}}}$ is a tensorial function that maps the space of invertible two-point tensors into the space of symmetric Lagrangian tensors. The relation between ${}^t\underline{\underline{\mathbf{m}}}$ (in Example 5.1) and ${}^t_\circ\underline{\underline{\mathbf{M}}}$ can be derived using Eq. (3.19).

For the fulfillment of the principle of equipresence, we make the same comment as in Example 5.1. To study the objectivity of the formulation, as usual, we consider two Cartesian systems in the spatial configuration: a fixed one $\{{}^t z_\alpha\}$ and a moving one $\{{}^t z^*_\alpha\}$.

Since the second Piola-Kirchhoff stress tensor is an objective Lagrangian tensor, we have:

$$ {}^t_\circ\underline{\underline{\mathbf{S}}} = {}^t_\circ\underline{\underline{\mathbf{S}}}^* . $$

From the objectivity of the deformation gradient tensor it follows that (Eq. (2.102c)),

$$ {}^t_\circ\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{Q}}}(t) \cdot {}^t_\circ\underline{\underline{\mathbf{X}}}^* $$

where $\underline{\underline{\mathbf{Q}}}(t)$ is an orthogonal tensor.

Due to material objectivity, the observer in the moving frame writes:

$$ {}^t_\circ\underline{\underline{\mathbf{S}}}^* = {}^t_\circ\underline{\underline{\mathbf{M}}}\left({}^t_\circ\underline{\underline{\mathbf{X}}}^*\right) $$

and particularizing for $\underline{\underline{\mathbf{Q}}}(t) = {}^t_\circ\underline{\underline{\mathbf{R}}}$ (from the polar decomposition ${}^t_\circ\underline{\underline{\mathbf{X}}} = {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}$),

$$ {}^t_\circ\underline{\underline{\mathbf{S}}}^* = {}^t_\circ\underline{\underline{\mathbf{M}}}\left({}^t_\circ\underline{\underline{\mathbf{R}}}^T \cdot {}^t_\circ\underline{\underline{\mathbf{R}}} \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}\right) = {}^t_\circ\underline{\underline{\mathbf{S}}}. $$

Hence,

$$ {}^t_\circ\underline{\underline{\mathbf{S}}}^* = {}^t_\circ\underline{\underline{\mathbf{M}}}\left({}^t_\circ\underline{\underline{\mathbf{U}}}\right) , $$

and using

$$ {}^t_\circ\underline{\underline{\mathbf{U}}} = \left[2\,{}^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}} + \underline{\underline{\mathbf{I}}}\right]^{1/2} $$

we get,

$$ {}^t_\circ\underline{\underline{\mathbf{S}}} = {}^t_\circ\underline{\underline{\widehat{\mathbf{M}}}}\left({}^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}\right) . $$

The principle of material-frame indifference imposes the above form of the constitutive relation. ◀◀◀◀◀

5.2 Constitutive relations in solid mechanics: purely mechanical formulations

In this section we will analyze some of the constitutive relations that are used to model the mechanical behavior of solids, neglecting the couplings with other physical phenomena.

An *elastic material model* (also called *Cauchy elastic material* (Ogden 1984)) predicts a material behavior independent of the material history and time, that is to say, stresses are univocally determined by strains and vice versa.

Of course, an elastic material model cannot be used to model: permanent deformation phenomena, damage of materials, creep effects, strain-rate effects, etc.

In Chap. 3 we presented the definition of conjugate stress and strain rate measures. If we define an arbitrary stress measure $\underline{\underline{\mathbf{T}}}$ (it can be either a Lagrangian, Eulerian or two-point tensor) and its conjugate strain-rate measure $\dot{\underline{\underline{\mathbf{E}}}}$, we can write the stress power per unit volume as,

$$P_\sigma = \underline{\underline{\mathbf{T}}}^T : \dot{\underline{\underline{\mathbf{E}}}} , \tag{5.1a}$$

for an elastic solid,

$$\underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}(\underline{\underline{\mathbf{E}}}) , \tag{5.1b}$$

hence,

$$P_\sigma = \underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) : \dot{\underline{\underline{\mathbf{E}}}} . \tag{5.1c}$$

In general (Ogden 1984), we cannot assess on the existence of a scalar function, $U(\underline{\underline{\mathbf{E}}})$, such that,

$$\dot{U} = \underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) : \dot{\underline{\underline{\mathbf{E}}}} , \tag{5.1d}$$

that is to say, in general P_σ is not an exact differential.

When P_σ is an exact differential, we can write (1944)

$$\dot{U} = \frac{\partial U}{\partial \underline{\underline{\mathbf{E}}}} : \dot{\underline{\underline{\mathbf{E}}}} = \underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) : \dot{\underline{\underline{\mathbf{E}}}} , \tag{5.1e}$$

and we say that the material model is hyperelastic (also called *Green elastic material* model (Ogden 1984)).

For a *hyperelastic material model*, from Eq. (5.1e) taking into account that $\dot{\underline{\underline{\mathbf{E}}}}$ is arbitrary,

$$\underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) = \frac{\partial U}{\partial \underline{\underline{\mathbf{E}}}} , \tag{5.1f}$$

and U is called the *elastic energy function per unit volume.*

Example 5.4. ◀◀◀◀◀

For a hyperelastic material,

$$T^{ij} = \frac{\partial U}{\partial E_{ij}} \qquad ; \qquad T^{kl} = \frac{\partial U}{\partial E_{kl}}$$

and since

$$\frac{\partial^2 U}{\partial E_{ij}\, \partial E_{kl}} = \frac{\partial^2 U}{\partial E_{kl}\, \partial E_{ij}} \ ,$$

we must have

$$\frac{\partial T^{ij}}{\partial E_{kl}} = \frac{\partial T^{kl}}{\partial E_{ij}} \ .$$

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The *inelastic mechanical behavior* of some materials can be described with equations of the form,

$$\mathbf{d}\underline{\underline{\mathbf{T}}} \ = \ \underline{\underline{\underline{\underline{\mathbf{C}}}}}(\underline{\underline{\mathbf{T}}} \ , \ \underline{\underline{\mathbf{E}}}) \ : \ \mathbf{d}\underline{\underline{\mathbf{E}}} \ , \tag{5.2}$$

which are the *hypoelastic material models*.

5.2.1 Hyperelastic material models

For a hyperelastic material, the elastic energy in the spatial configuration per unit volume of the reference configuration can be written as:

$$\mathrm{d}\ {}^t_\circ\mathtt{U} \ = \ {}^t_\circ S^{IJ} \ \mathrm{d}\ {}^t_\circ\varepsilon_{IJ} \ , \tag{5.3a}$$

using the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor.

If we use ${}^t\mathtt{U}$ as the elastic energy in the spatial configuration per unit mass, we get

$$\mathrm{d}\ {}^t_\circ\mathtt{U}\ \frac{dm}{{}^\circ\rho} \ = \ \mathrm{d}{}^t\mathtt{U}\ dm \ . \tag{5.3b}$$

Hence,

$$\mathrm{d}{}^t\mathtt{U} \ = \ \frac{1}{{}^\circ\rho}\ {}^t_\circ S^{IJ} \ \mathrm{d}\ {}^t_\circ\varepsilon_{IJ} \ , \tag{5.3c}$$

and

$${}^t_\circ S^{IJ} \ = \ {}^\circ\rho\ \frac{\partial^t\mathtt{U}}{\partial^t_\circ\varepsilon_{IJ}} \ = \ 2\ {}^\circ\rho\ \frac{\partial^t\mathtt{U}}{\partial^t_\circ C_{IJ}} \ . \tag{5.3d}$$

Using the chain rule and considering Eq. (4.51a), we can write (Marsden & Hughes 1983)

$$\frac{\partial^t\mathtt{u}}{\partial^t g_{ab}} \ = \ \frac{\partial^t\mathtt{U}}{\partial^t_\circ C_{AB}}\ \frac{\partial^t_\circ C_{AB}}{\partial^t g_{ab}} \ , \tag{5.4a}$$

from Eq. (2.93a)

$$ {}^t_\circ C_{AB} \;=\; {}^t g_{ab}\ {}^t_\circ X^a{}_A\ {}^t_\circ X^b{}_B\ , \tag{5.4b}$$

and therefore

$$ \frac{\partial\, {}^t\mathtt{u}}{\partial\, {}^t g_{ab}} \;=\; \frac{\partial\, {}^t\mathtt{U}}{\partial\, {}^t_\circ C_{AB}}\,{}^t_\circ X^a{}_A\ {}^t_\circ X^b{}_B \;=\; {}^t\phi_*\Big(\frac{\partial\, {}^t\mathtt{U}}{\partial\, {}^t_\circ C_{AB}}\Big)\ . \tag{5.4c}$$

Pushing-forward Eq. (5.3d) and using the above, we obtain

$$ {}^t\tau^{ij} \;=\; 2\ {}^\circ\rho\ \frac{\partial\, {}^t\mathtt{u}}{\partial\, {}^t g_{ij}}\ , \tag{5.4d}$$

and using Eq. (3.12) we obtain the Doyle-Ericksen formula (Simo & Marsden 1984):

$$ {}^t\sigma^{ij} \;=\; 2\ {}^t\rho\ \frac{\partial\, {}^t\mathtt{u}}{\partial\, {}^t g_{ij}}\ . \tag{5.4e}$$

It is important to realize that the above is the correct relation for deriving the Cauchy stress tensor from the elastic energy per unit mass function, and that (Marsden & Hughes 1983):

$$ {}^t\sigma^{ij} \;\neq\; 2\ {}^t\rho\ \frac{\partial\, {}^t\mathtt{u}}{\partial\, {}^t e_{ij}}\ . \tag{5.5}$$

Using the result in the Example 4.16 and remembering that for a hyperelastic material the stress is only a function of the state variables, we also get

$$ {}^t_\circ P_a{}^{\ B} \;=\; {}^\circ\rho\ \frac{\partial\, {}^t\mathtt{U}}{\partial\, {}^t_\circ X^a{}_B}\ .$$

5.2.2 A simple hyperelastic material model

In this section we will develop the simplest possible hyperelastic material model: *the isotropic, linear hyperelastic material model.*

To use Eq. (5.3d), in the reference configuration we need to define the strain energy per unit volume of the reference configuration

$$ {}^t_\circ\mathtt{U} \;=\; {}^t_\circ\mathtt{U}({}^t_\circ\varepsilon_{IJ})\ , \tag{5.6a}$$

the simplest definition is:

$$ {}^t_\circ\mathtt{U} \;=\; {}^t_\circ A_\circ \;+\; {}^t_\circ B^{IJ}\ {}^t_\circ\varepsilon_{IJ} \;+\frac{1}{2}\ {}^t_\circ\hat{C}^{IJKL}\ {}^t_\circ\varepsilon_{IJ}\ {}^t_\circ\varepsilon_{KL}\ , \tag{5.6b}$$

where ${}^t_\circ A_\circ$; ${}^t_\circ B^{IJ}$ and ${}^t_\circ\hat{C}^{IJKL}$ are constants (independent of the deformation).

- The value of ${}^{t}_{\circ}A_{\circ}$ is arbitrary since we are only interested in the derivatives of ${}^{t}_{\circ}\mathtt{U}$, therefore we set

$$ {}^{t}_{\circ}A_{\circ} \;=\; 0 \qquad . \tag{5.6c}$$

- Using Eq. (5.1f) with Eq. (5.6b) we get,

$$ {}^{t}_{\circ}S^{IJ} \;=\; {}^{t}_{\circ}B^{IJ} \;+\frac{1}{2}\,\left({}^{t}_{\circ}\hat{C}^{IJKL} + {}^{t}_{\circ}\hat{C}^{KLIJ}\right)\,{}^{t}_{\circ}\varepsilon_{KL} \qquad . \tag{5.6d}$$

Since we are not considering an initial stress/strain state, ${}^{t}_{\circ}\varepsilon_{IJ} \;=\; 0 \iff {}^{t}_{\circ}S^{IJ} \;=\; 0$ and therefore, we must have

$$ {}^{t}_{\circ}B^{IJ} \;=\; 0\;. \tag{5.6e}$$

Hence, for the simplest hyperelastic material model, we have the elastic energy per unit volume of the reference configuration expressed as a quadratic form of the Green-Lagrange strain tensor;

$$ {}^{t}_{\circ}\mathtt{U} \;=\; \frac{1}{2}\,{}^{t}_{\circ}\hat{C}^{IJKL}\,{}^{t}_{\circ}\varepsilon_{IJ}\,{}^{t}_{\circ}\varepsilon_{KL} \tag{5.7a}$$

and a linear stress/strain relation,

$$ {}^{t}_{\circ}S^{IJ} \;=\; {}^{t}_{\circ}C^{IJKL}\,{}^{t}_{\circ}\varepsilon_{KL} \tag{5.7b}$$

where (Malvern 1969),

$$ {}^{t}_{\circ}C^{IJKL} = \frac{1}{2}\,\left({}^{t}_{\circ}\hat{C}^{IJKL} + {}^{t}_{\circ}\hat{C}^{KLIJ}\right)\;. \tag{5.7c}$$

Doing a push-forward of the above equation to the spatial configuration we get,

$$ {}^{t}\tau^{ij} \;=\; {}^{t}c^{ijkl}\,{}^{t}e_{kl} \tag{5.8a}$$

where the *spatial elasticity tensor* is defined by

$$ {}^{t}c^{ijkl} \;=\; {}^{t}_{\circ}C^{IJKL}\,{}^{t}_{\circ}X^{i}{}_{I}\,{}^{t}_{\circ}X^{j}{}_{J}\,{}^{t}_{\circ}X^{k}{}_{K}\,{}^{t}_{\circ}X^{l}{}_{L}\;. \tag{5.8b}$$

Note that the obtained spatial elasticity tensor has components that are a function of the deformation (not constant).

The Second Law of Thermodynamics indicates that for deforming a real (stable) material we must spend an amount of work; hence,

$$ {}^{t}_{\circ}\mathtt{U} \;\geq\; 0 \tag{5.9}$$

and we can only have ${}^{t}_{\circ}\mathtt{U} \;=\; 0$ when ${}^{t}_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}} \;=\; \underline{\underline{\mathbf{0}}}$.

Therefore, ${}^{t}_{\circ}\mathtt{U}$ is a positive definite quadratic form.

Using Eqs. (5.7b) and (5.8a) together with the quotient rule (Sect. A.5) we realize that ${}^{t}_{\circ}C^{IJKL}$ and ${}^{t}c^{ijkl}$ are components of two fourth-order tensors, ${}^{t}_{\circ}\underline{\underline{\underline{\underline{\mathbf{C}}}}}$ and ${}^{t}\underline{\underline{\underline{\underline{\mathbf{c}}}}}$.

► Symmetries of ${}^{t}_{\circ}\underline{\underline{\underline{\underline{\mathbf{C}}}}}$

In Eq. (5.7b) ${}^{t}_{\circ}S^{IJ}$ and ${}^{t}_{\circ}\varepsilon^{IJ}$ are components of symmetric tensors; hence,

$$ {}^{t}_{\circ}C^{IJKL} = {}^{t}_{\circ}C^{JIKL} \tag{5.10a}$$

$$ {}^{t}_{\circ}C^{IJKL} = {}^{t}_{\circ}C^{IJLK} \quad . \tag{5.10b}$$

Therefore the original 81 components of ${}^{t}_{\circ}\underline{\underline{\underline{\underline{\mathbf{C}}}}}$ are reduced to only 36 independent ones.

Using for this case the result in Example 5.4 we arrive at,

$$ {}^{t}_{\circ}C^{IJKL} = {}^{t}_{\circ}C^{KLIJ} \ . \tag{5.11}$$

Therefore, we are left with only 21 independent constants.

It is important to note that while Eqs. (5.10a-5.10b) apply to any material, Eq. (5.11) is only valid for a hyperelastic material.

> Without introducing any symmetry inherent to a particular material model, for the description of the most general linear hyperelastic material model we have to use 21 material constants.

Now we will consider materials with inherent symmetries, that is to say, with *planes of elastic symmetry.* Let us first consider a material with one plane of elastic symmetry; this means that if we define two coordinate systems $\{{}^{\circ}x^{I}\}$ and $\{{}^{\circ}\tilde{x}^{I}\}$ with:

- ${}^{\circ}x^{1}$ and ${}^{\circ}x^{2}$ on the symmetry plane,
- ${}^{\circ}x^{3}$ normal to the symmetry plane,
- ${}^{\circ}\tilde{x}^{1} = {}^{\circ}x^{1}$; ${}^{\circ}\tilde{x}^{2} = {}^{\circ}x^{2}$ and ${}^{\circ}\tilde{x}^{3} = -\ {}^{\circ}x^{3}$,

we must have

$$ {}^{t}_{\circ}C^{IJKL} = {}^{t}_{\circ}\tilde{C}^{IJKL} \quad . \tag{5.12}$$

Taking into account the tensor transformations

$$ {}^{t}_{\circ}\tilde{C}^{IJKL} = {}^{t}_{\circ}C^{PQRS} \frac{\partial^{\circ}\tilde{x}^{I}}{\partial^{\circ}x^{P}} \frac{\partial^{\circ}\tilde{x}^{J}}{\partial^{\circ}x^{Q}} \frac{\partial^{\circ}\tilde{x}^{K}}{\partial^{\circ}x^{R}} \frac{\partial^{\circ}\tilde{x}^{L}}{\partial^{\circ}x^{S}} \tag{5.13}$$

it is obvious that the fulfillment of Eq. (5.12) imposes that the components with an odd quantity of indices “3” have to be zero. Hence, a plane of elastic symmetry reduces the material constants from 21 to 13.

Now we consider a further simplification imposed by the consideration of more elastic symmetries: the *orthotropic* hyperelastic material model. In this case we have to consider three mutually orthogonal planes of elastic symmetry.

The intersection of the three orthotropy planes determines a Cartesian coordinate system $\{{}^{\circ}z^{\alpha}\}$.

Besides the symmetry considerations with respect to the $({}^{\circ}z_1\ ,\ {}^{\circ}z_2)$ plane, derived above, we have to consider a symmetry with respect to the $({}^{\circ}z_2\ ,\ {}^{\circ}z_3)$ plane (Sokolnikoff 1956).

The fulfillment of Eqs. (5.12) and (5.13) imposes, for this second symmetry, a further 4 zero material constants; hence, an orthotropic material model has only 9 material constants.

Examples of materials that are adequately described using orthotropic material models are wood, rolled steel plates, etc.

It is interesting to examine further the orthotropic constitutive relation using the following arrays:

$$\begin{bmatrix} {}^t_{\circ}S^{11} \\ {}^t_{\circ}S^{22} \\ {}^t_{\circ}S^{33} \\ {}^t_{\circ}S^{12} \\ {}^t_{\circ}S^{23} \\ {}^t_{\circ}S^{31} \end{bmatrix} = \begin{bmatrix} {}^t_{\circ}C^{1111} & {}^t_{\circ}C^{1122} & {}^t_{\circ}C^{1133} & 0 & 0 & 0 \\ & {}^t_{\circ}C^{2222} & {}^t_{\circ}C^{2233} & 0 & 0 & 0 \\ & & {}^t_{\circ}C^{3333} & 0 & 0 & 0 \\ & & & {}^t_{\circ}C^{1212} & 0 & 0 \\ & & & & {}^t_{\circ}C^{2323} & 0 \\ sym & & & & & {}^t_{\circ}C^{3131} \end{bmatrix} \begin{bmatrix} {}^t_{\circ}\varepsilon_{11} \\ {}^t_{\circ}\varepsilon_{22} \\ {}^t_{\circ}\varepsilon_{33} \\ 2\ {}^t_{\circ}\varepsilon_{12} \\ 2\ {}^t_{\circ}\varepsilon_{23} \\ 2\ {}^t_{\circ}\varepsilon_{31} \end{bmatrix} \quad . \tag{5.14}$$

It is obvious from the above equation that if the orthotropy axes are coincident with the principal axes of strain, they are also the principal axes of the stress tensor (collinearity between the stress and strain tensor).

When for an orthotropic material the constitutive relation is written in a coordinate system that is not coincident with the material orthotropy system, the convenient form of Eq. (5.14) is lost.

▶ The isotropic hyperelastic material model

An important case of materials with inherent symmetry is analyzed in this section: when every plane is a plane of elastic symmetry we have an *isotropic* material; therefore ${}^t_{\circ}\underline{\underline{\underline{\underline{\mathbf{C}}}}}$ is an isotropic fourth-order tensor (Aris 1962).

It was shown in (Aris 1962) that the most general fourth-order isotropic tensor has Cartesian components of the form,

$${}^t_{\circ}C_{\alpha\beta\gamma\delta} \;=\; \lambda\, \delta_{\alpha\beta}\, \delta_{\gamma\delta} + \mu\, (\, \delta_{\alpha\gamma}\, \delta_{\beta\delta} + \delta_{\alpha\delta}\, \delta_{\beta\gamma}\,) + \tau\, (\, \delta_{\alpha\gamma}\, \delta_{\beta\delta} - \delta_{\alpha\delta}\, \delta_{\beta\gamma}\,) \quad . \tag{5.15}$$

Any of the two Eqs. (5.10a-5.10b), derived from the symmetry of the stress and strain tensors, imposes the condition:

$$\tau \;=\; 0 \quad . \tag{5.16}$$

Hence, it is obvious from Eqs. (5.15) and (5.16) that in the case of isotropic materials there are only 2 independent constants; usually (Malvern 1969):

- E : Young's modulus
- ν : Poisson's ratio.

For *any* Cartesian system we can write Eq. (5.14) as, (Bathe 1996):

$$\begin{bmatrix} {}_{\circ}^{t}S^{11} \\ {}_{\circ}^{t}S^{22} \\ {}_{\circ}^{t}S^{33} \\ {}_{\circ}^{t}S^{12} \\ {}_{\circ}^{t}S^{23} \\ {}_{\circ}^{t}S^{31} \end{bmatrix} = \frac{E\ (1-\nu)}{(1+\nu)\ (1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ sym & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} {}_{\circ}^{t}\varepsilon_{11} \\ {}_{\circ}^{t}\varepsilon_{22} \\ {}_{\circ}^{t}\varepsilon_{33} \\ 2\ {}_{\circ}^{t}\varepsilon_{12} \\ 2\ {}_{\circ}^{t}\varepsilon_{23} \\ 2\ {}_{\circ}^{t}\varepsilon_{31} \end{bmatrix}, \tag{5.17}$$

and the stress and strain tensors are always collinear.

It is important to realize that the numerical values of E and ν usually found in the engineering literature refer to the relation [*Cauchy stresses/infinitesimal strains*] rather than to the relation [*second Piola-Kirchhoff stresses/Green-Lagrange strains*] that we use in this section.

We can define K , the volumetric modulus and G , the shear modulus as:

$$K = \frac{E}{3\ (1-2\nu)} \tag{5.18a}$$

$$G = \frac{E}{2\ (1+\nu)} \tag{5.18b}$$

and write for the simplest hyperelastic material model that we consider in this section,

$${}_{\circ}^{t}S_{\alpha\beta} = K\ {}_{\circ}^{t}\varepsilon_{V}\ \delta_{\alpha\beta} + 2\ G\ {}_{\circ}^{t}\varepsilon'_{\alpha\beta} \tag{5.19a}$$

$${}_{\circ}^{t}\mathtt{U} = \frac{1}{2}\ K\ {}_{\circ}^{t}\varepsilon_{V}{}^{2} + G\ {}_{\circ}^{t}\varepsilon'_{\alpha\beta}\ {}_{\circ}^{t}\varepsilon'_{\alpha\beta} \tag{5.19b}$$

where, ${}_{\circ}^{t}\varepsilon_{V}$ is the volumetric or hydrostatic component or the Green-Lagrange strain tensor, ${}_{\circ}^{t}\varepsilon_{V} = {}_{\circ}^{t}\varepsilon_{\alpha\beta}\ \delta_{\alpha\beta}$, and ${}_{\circ}^{t}\varepsilon'_{\alpha\beta}]$ is the deviatoric component of the Green-Lagrange strain tensor, ${}_{\circ}^{t}\varepsilon'_{\alpha\beta} = {}_{\circ}^{t}\varepsilon_{\alpha\beta} - \frac{1}{3}\ \delta_{\alpha\beta}\ {}_{\circ}^{t}\varepsilon_{V}$.

From Eq. (5.19a-5.19b) we conclude that:

$$K \geq 0 \tag{5.20a}$$

$$G \geq 0\quad . \tag{5.20b}$$

It is important to realize that ${}_{\circ}^{t}\varepsilon_V$ is not a measure of the continuum volume change.

The above implies,

$$E \geq 0 \tag{5.20c}$$

$$-1 \leq \nu \leq 0.5 \quad . \tag{5.20d}$$

We must remember that the symmetries discussed in the last two Sections are *directional* properties and *not positional* properties. Even if a material has a certain elastic symmetry at each point, the properties may vary from point to point in a manner not possessing any symmetry with respect to the shape of the analyzed body (Malvern 1969).

When considering an isotropic material ${}_{\circ}^{t}\underline{\underline{\mathbf{S}}}$ and ${}_{\circ}^{t}\underline{\underline{\boldsymbol{\varepsilon}}}$ are always collinear tensors.

Example 5.5. ◀◀◀◀◀

In the above equations we expressed the constitutive tensor for orthotropic and isotropic material in Cartesian coordinates.

To express it in a general coordinate system $\{{}^{\circ}x^{I}\}$ we use:

$$ {}_{\circ}^{t}\tilde{C}^{IJKL} = {}_{\circ}^{t}C^{\alpha\beta\gamma\delta} \left(\underline{\mathbf{e}}_{\alpha} \cdot {}^{\circ}\underline{\mathbf{g}}^{I}\right) \left(\underline{\mathbf{e}}_{\beta} \cdot {}^{\circ}\underline{\mathbf{g}}^{J}\right) \left(\underline{\mathbf{e}}_{\gamma} \cdot {}^{\circ}\underline{\mathbf{g}}^{K}\right) \left(\underline{\mathbf{e}}_{\delta} \cdot {}^{\circ}\underline{\mathbf{g}}^{L}\right) \ . $$

Even though the material is still defined by the same number of material constants, the convenient form of Eqs. (5.14) and (5.17) is lost. ◀◀◀◀◀

Example 5.6. ◀◀◀◀◀

A hyperelastic constitutive model cannot be formulated, in the spatial configuration, as

$$ {}^{t}\sigma^{ij} = {}^{t}c^{ijkl}\ {}^{t}e_{kl} \tag{A}$$

where ${}^{t}\underline{\underline{\underline{\underline{\mathbf{c}}}}}$ is a constant and isotropic fourth order tensor

If Eq. (A) is an acceptable constitutive relation, then taking Lie derivatives on both sides of the equal sign, we get for a hyperelastic material,

$$ {}^{t}\overset{\circ}{\sigma}{}^{ij} = {}^{t}c^{ijkl}\ {}^{t}d_{kl} \ . $$

In Example 5.14, we will show that the above equation cannot represent a hyperelastic material; hence, Eq. (A) cannot either.

It is important to point out that in (Simo & Pister 1984) it was also demonstrated that the spatial elasticity tensor cannot be an isotropic and constant fourth-order tensor. Simo and Pister developed their demonstration without the need of going through an equivalent hypoelastic material; however, we could not reproduce the demonstration in (Simo & Pister 1984). ◀◀◀◀◀

5.2.3 Other simple hyperelastic material models

In Sect. 5.2.1 we have formulated a simple hyperelastic material model using:

$$ {}_{\circ}^{t}\mathrm{U} = \frac{1}{2}\, {}_{\circ}^{t}\underline{\underline{\boldsymbol{\varepsilon}}} : {}_{\circ}^{t}\underline{\underline{\underline{\underline{\mathbf{C}}}}} : {}_{\circ}^{t}\underline{\underline{\boldsymbol{\varepsilon}}} \tag{5.21a} $$

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{S}}} = \frac{\partial {}_{\circ}^{t}\mathrm{U}}{\partial {}_{\circ}^{t}\underline{\underline{\boldsymbol{\varepsilon}}}} \; . \tag{5.21b} $$

We can use other conjugate stress/strain rate measures and get, using for the elastic energy per unit volume in the reference configuration quadratic forms with constant coefficients:

$$ {}_{\circ}^{t}\mathrm{U} = \frac{1}{2}\, {}_{\circ}^{t}\underline{\underline{\mathbf{H}}} : {}_{\circ}^{t}\underline{\underline{\underline{\underline{\mathbf{C}}}}} : {}_{\circ}^{t}\underline{\underline{\mathbf{H}}} \tag{5.22a} $$

$$ {}^{t}\underline{\underline{\mathbf{\Gamma}}} = \frac{\partial {}_{\circ}^{t}\mathrm{U}}{\partial {}_{\circ}^{t}\underline{\underline{\mathbf{H}}}} \; . \tag{5.22b} $$

As we saw in Sect. 3.3.4, Eqs. (5.22a-5.22b) can only be used with isotropic materials.

Even considering the above limitation, Eqs. (5.22a-5.22b) are very useful for formulating practical constitutive relations because it was experimentally shown by Anand that for metals, the values of E and ν that are used in the usual engineering environment [*Cauchy stresses/infinitesimal strains*] can be accurately used in Eqs. (5.22a-5.22b) for quite large strain values (Anand 1979).

Also, for the case of infinitesimal strains,

$$ {}_{\circ}^{t}\mathrm{U} = \frac{1}{2}\, {}^{t}\underline{\underline{\boldsymbol{\varepsilon}}} : {}_{\circ}^{t}\underline{\underline{\underline{\underline{\mathbf{C}}}}} : {}^{t}\underline{\underline{\boldsymbol{\varepsilon}}} \; , \tag{5.23a} $$

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = \frac{\partial {}_{\circ}^{t}\mathrm{U}}{\partial\, {}^{t}\underline{\underline{\boldsymbol{\varepsilon}}}} \; , \tag{5.23b} $$

$$ {}^{t}\underline{\underline{\boldsymbol{\tau}}} = \left(\frac{{}^{\circ}\rho}{{}^{t}\rho}\right) \frac{\partial {}_{\circ}^{t}\mathrm{U}}{\partial\, {}^{t}\underline{\underline{\boldsymbol{\varepsilon}}}} \; , \tag{5.23c} $$

where ${}^{t}\underline{\underline{\boldsymbol{\varepsilon}}}$ is the infinitesimal strain tensor defined in Example 2.26. In this case since ${}^{t}\rho \approx {}^{\circ}\rho$, ${}^{t}\underline{\underline{\boldsymbol{\sigma}}} \approx {}^{t}\underline{\underline{\boldsymbol{\tau}}}$.

The above equations are possible because

$$ \frac{D\, {}^{t}\underline{\underline{\boldsymbol{\varepsilon}}}}{Dt} = {}^{t}\underline{\underline{\mathbf{d}}} \; . \tag{5.24} $$

As we have seen above, in Eq. (5.5), it is not possible to use an alternative hyperelastic formulation in terms of the Almansi strain tensor.

5.2.4 Ogden hyperelastic material models

For an isotropic material we can write,

$$ {}^{t}_{\circ}\mathtt{U} = {}^{t}_{\circ}\mathtt{U}(\lambda_1, \lambda_2, \lambda_3) \tag{5.25}$$

where the λ_i are the eigenvalues of the second order tensor, ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$, that is to say, the principal stretches defined in Eq. (2.58e).

Since

$$ \lambda_i = \lambda_i(I_1^C, I_2^C, I_3^C) \tag{5.26}$$

where the values (I_1^C, I_2^C, I_3^C) are the invariants of ${}^{t}_{\circ}\underline{\underline{\mathbf{C}}}$ defined in Eqs. (2.59b-2.59d); we can write

$$ {}^{t}_{\circ}\mathtt{U} = {}^{t}_{\circ}\mathtt{U}(I_1^C, I_2^C, I_3^C) \tag{5.27a}$$

$$ I_1^C = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 \tag{5.27b}$$

$$ I_2^C = (\lambda_2)^2 (\lambda_3)^2 + (\lambda_3)^2 (\lambda_1)^2 + (\lambda_1)^2 (\lambda_2)^2 \tag{5.27c}$$

$$ I_3^C = (\lambda_1)^2 (\lambda_2)^2 (\lambda_3)^2 \ . \tag{5.27d}$$

Ogden proposed to write ${}^{t}_{\circ}\mathtt{U}(I_1^C, I_2^C, I_3^C)$ as an infinite series in powers of $\left(I_1^C - 3\right), \left(I_2^C - 3\right), \left(I_3^C - 1\right)$ (Ogden 1984).

Thus,

$$ {}^{t}_{\circ}\mathtt{U}(\lambda_1, \lambda_2, \lambda_3) = \sum_{p,q,r=0}^{\infty} C_{pqr} \left(I_1^C - 3\right)^p \ \left(I_2^C - 3\right)^q \ \left(I_3^C - 1\right)^r \tag{5.28}$$

where the coefficients C_{pqr} are independent of the deformation.

For the unstrained configuration,

$$ \lambda_1 = \lambda_2 = \lambda_3 = 1 $$

$$ I_1^C = 3 $$

$$ I_2^C = 3 $$

$$ I_3^C = 1 \ . $$

Therefore, in the unstrained configuration the strain energy per unit mass is zero provided that $C_{000} = 0$.

Also, the unstrained configuration has to be stress free; hence,

$$ \frac{\partial {}^{t}_{\circ}\mathtt{U}}{\partial \lambda_i} \Big|_{I_1^C = I_2^C = 3; I_3^C = 1} = 0 \qquad for \ \ i = 1, 2, 3 \ . \tag{5.29a}$$

The above equation leads using Eq. (5.28), to the condition,

$$ C_{100} + 2\ C_{010} + C_{001} = 0 \ . \tag{5.29b}$$

Example 5.7. ◀◀◀◀◀

The third invariant, I_3^C , has an important physical interpretation. Considering the eigendirections of ${}^t_\circ\underline{\underline{\mathbf{U}}}$, a differential volume ${}^\circ\mathrm{d}V$ defined in the reference configuration along those directions is transformed, in the spatial configuration, in a differential volume ${}^t\mathrm{d}V$:

$$ {}^t\mathrm{d}V = \lambda_1\lambda_2\lambda_3 \; {}^\circ\mathrm{d}V $$

hence,

$$ {}^tJ \;=\; \frac{{}^t\mathrm{d}V}{{}^\circ\mathrm{d}V} = \sqrt{I_3^C} \;. $$

That is to say, the invariant I_3^C describes the volume change during the continuum deformation.
Obviously, for incompressible deformations,

$$ {}^tJ \;= I_3^C = 1 \;. $$

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Here Ogden introduces a simplificative hypothesis: the strain energy is decoupled in two parts, a deviatoric part that is independent of the volume change and a volumetric part that is only a function of the volume change (Ogden 1984). The above hypothesis is introduced in Eq. (5.28) by imposing,

$$ \begin{aligned} {}^t_\circ\mathtt{U}(\lambda_1,\lambda_2,\lambda_3) = & \sum_{p,q=0}^{\infty} C_{pq0}\left(I_1^C-3\right)^p \; \left(I_2^C-3\right)^q \\ & + \sum_{r=1}^{\infty} C_{00r}\left(I_3^C-1\right)^r \;. \end{aligned} \tag{5.30} $$

We can state,

$$ g(I_3^C) = \sum_{r=1}^{\infty} C_{00r}\left(I_3^C-1\right)^r \tag{5.31} $$

and retain only a few terms in Eq. (5.30); in this way we obtain specific hyperelastic relations that have been successfully used for particular materials.

For example, for *incompressible materials*:

$$ I_3^C = 1 \tag{5.32a} $$

$$ g(I_3^C) = 0 \tag{5.32b} $$

$$ {}^t_\circ\mathtt{U} = \sum_{p,q=0}^{\infty} C_{pq0}\left(I_1^C-3\right)^p \; \left(I_2^C-3\right)^q \;. \tag{5.32c} $$

- Retaining only two terms in Eq. (5.32c) we obtain the *Mooney-Rivlin* strain energy function that has been successfully used for rubber-like materials,

$$ {}^t_\circ\mathsf{U} = C_{100}\left(I_1^C - 3\right) + C_{010}\ \left(I_2^C - 3\right)\ . \tag{5.33}$$

- If $C_{010} = 0$ the above reduces to the *neo-Hookean* strain energy function which has played an important role in the development of the theory and applications of nonlinear elasticity,

$$ {}^t_\circ\mathsf{U} = C_{100}\left(I_1^C - 3\right)\ . \tag{5.34}$$

The principal values of the collinear Green-Lagrange and second Piola-Kirchhoff tensors are related by,

$$ {}^t_\circ S_i = \frac{\partial\ {}^t_\circ\mathsf{U}}{\partial\varepsilon_i} = \frac{\partial\ {}^t_\circ\mathsf{U}}{\partial\lambda_j}\frac{\partial\lambda_j}{\partial\varepsilon_i} \tag{5.35a}$$

taking into account that the ε_i are the eigenvalues of the Green-Lagrange strain tensor, we can write

$$ \frac{\partial\lambda_j}{\partial\varepsilon_i} = \frac{1}{\lambda_j}\delta_{ji} \qquad \text{(no addition on } j)\ . \tag{5.35b}$$

Hence,

$$ {}^t_\circ S_i = \frac{1}{\lambda_i}\frac{\partial {}^t_\circ\mathsf{U}}{\partial\lambda_i} \qquad \text{(no addition on } i)\ . \tag{5.35c}$$

With a push-forward of the above, we get the eigenvalues of the Kirchhoff stress tensor,

$$ {}^t\tau_i = \lambda_i\frac{\partial {}^t_\circ\mathsf{U}}{\partial\lambda_i} \qquad \text{(no addition on } i)\ . \tag{5.35d}$$

Hence, the eigenvalues of the Cauchy stress tensor are,

$$ {}^t\sigma_i = \left(\frac{{}^t\rho}{{}^\circ\rho}\right)\lambda_i\frac{\partial {}^t_\circ\mathsf{U}}{\partial\lambda_i} \qquad \text{(no addition on } i)\ . \tag{5.35e}$$

When the material is incompressible the three values of λ_i are not independent since they have to fulfill the relation,

$$ {}^tJ - 1 = \lambda_1\lambda_2\lambda_3 - 1 = 0\ . \tag{5.36a}$$

In this case, Eq. (5.35a) determines the eigenvalues of the Cauchy stress tensor except for the value of the hydrostatic pressure, $\triangle P$:

$$ {}^t\sigma_i = \lambda_i\frac{\partial {}^t_\circ\mathsf{U}}{\partial\lambda_i} + \triangle P \qquad \text{(no addition on } i)\ . \tag{5.36b}$$

For determining the value of the hydrostatic pressure the equilibrium equations shall be used.

Example 5.8. ◀◀◀◀◀

In the case of the biaxial tension of a square incompressible sheet (Ogden 1984),

$$\lambda_1 \lambda_2 \lambda_3 = 1$$
$$\lambda_1 = \lambda_2 = \hat{\lambda}$$

therefore,

$$\lambda_3 = \frac{1}{(\hat{\lambda})^2} \ .$$

Also,

$${}^t\sigma_1 = {}^t\sigma_2 = {}^t\hat{\sigma}$$
$${}^t\sigma_3 = 0 \ .$$

Using the constitutive equation of the incompressible neo-Hookean material we get,

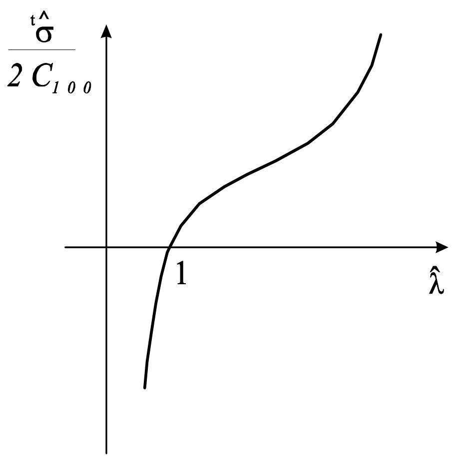
$${}^t_\circ\mathtt{U} = C_{100}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \ .$$

Equations (5.36b) give,

$${}^t\hat{\sigma} = 2C_{100}(\hat{\lambda})^2 + \triangle p$$
$$0 = 2C_{100}\frac{1}{(\hat{\lambda})^4} + \triangle p \ .$$

Hence,

$${}^t\hat{\sigma} = 2C_{100}\left[(\hat{\lambda})^2 - \frac{1}{(\hat{\lambda})^4}\right] \ .$$



In plane Cauchy stresses in the biaxial tension of a square incompressible sheet

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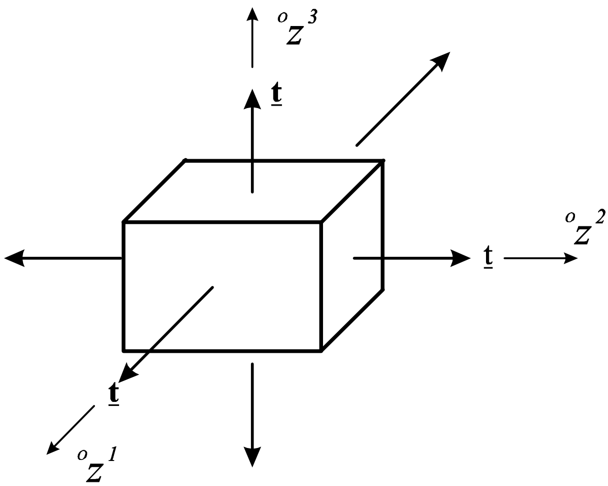
Example 5.9. ◀◀◀◀◀

For the case of an incompressible cube subjected to a uniform tension on its faces, in (Marsden & Hughes 1983) a neo-Hookean material is considered with a strain energy function of the form,

$$ {}^{t}_{\circ}\mathtt{U} = C_{100}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right] \ . $$



t *: force per unit area of the undeformed configuration*

Incompressible cube subjected to a uniform tension on its faces

Using Eqs. (5.36b), we get

$$ {}^{t}\sigma_i = 2C_{100}(\lambda_i)^2 + \triangle p \qquad (i = 1, 2, 3) \ . $$

From the equilibrium between internal and external forces, and taking into account that due to incompressibility

$$ \lambda_1\lambda_2\lambda_3 = 1 $$

we get,

$$ 2C_{100}(\lambda_1)^2 + \triangle p = t\ \lambda_1 \ , $$
$$ 2C_{100}(\lambda_2)^2 + \triangle p = t\ \lambda_2 \ , $$
$$ 2C_{100}(\lambda_3)^2 + \triangle p = t\ \lambda_3 \ , $$

where $t = |\underline{t}|$.

Eliminating $\triangle p$ gives,

$$\begin{aligned}
&[2C_{100}\,(\lambda_1+\lambda_2)-t]\,(\lambda_1-\lambda_2)=0\\
&[2C_{100}\,(\lambda_2+\lambda_3)-t]\,(\lambda_2-\lambda_3)=0\\
&[2C_{100}\,(\lambda_3+\lambda_1)-t]\,(\lambda_3-\lambda_1)=0\ .
\end{aligned}$$

In the intuitive case

$$\lambda_1=\lambda_2=\lambda_3$$

and the above equations are automatically fulfilled.
However, we will explore the possibility of other solutions.
• If we assume

$$\lambda_1\neq\lambda_2\neq\lambda_3$$

we must have

$$t=2C_{100}\,(\lambda_1+\lambda_2)=2C_{100}\,(\lambda_2+\lambda_3)=2C_{100}\,(\lambda_3+\lambda_1)\ \ .$$

Hence, we must have $\lambda_1=\lambda_2=\lambda_3$. This contradiction shows that the assumed solution is not possible.
• If we assume that two λ_is are equal and the third one is different, for example:

$$\lambda_1=\lambda_2\neq\lambda_3\ ,$$

we get

$$t=2C_{100}\,(\lambda_1+\lambda_3)$$

and due to incompressibility we must have,

$$2C_{100}\left[\lambda_1+\frac{1}{(\lambda_1)^2}\right]-t=0\ .$$

In order to obtain $\lambda_1=\lambda_1(t)$ we must solve the equation

$$f(\lambda)=\lambda_1^3-\frac{t}{2C_{100}}\lambda_1^2+1=0\ .$$

Considering that only the positive roots are admisible, it is shown in (Marsden & Hughes 1983) that:
☐ If $t<3\ \sqrt[3]{2}\ C_{100}\ \Longrightarrow$ no roots $\lambda_1>0$.
☐ If $t=3\ \sqrt[3]{2}\ C_{100}\ \Longrightarrow$ one root $\lambda_1>0$.
☐ If $t>3\ \sqrt[3]{2}\ C_{100}\ \Longrightarrow$ two positive roots $\lambda_1>0$.

Hence, $t=3\ \sqrt[3]{2}\ C_{100}$ is a bifurcation load. ________________ ◀◀◀◀◀

5.2.5 Elastoplastic material model under infinitesimal strains

In this section we are going to present the elastoplastic material model, considering only a purely mechanical formulation and infinitesimal strains.

In forthcoming sections we are going to present a thermoelastoplastic material model and we are going to extend the model to consider finite strains.

The elastoplastic material model has to describe the following experimentally observed phenomena:

- For loads below a certain *limit loading condition*, established via a *yield criterion*, the material behavior is elastic and can be described using the hyperelastic relations that we have discussed above.
- When the limit loading condition is reached there is an onset of *permanent or plastic deformations.*
- The plastic deformations produce an evolution of the yield condition that is described via a *hardening law.*
- When the limit loading condition is reached and then an *unloading* takes place, elastic deformations are developed.
- The material behavior is *rate independent*, that is to say, the material behavior is not a function of the loading or deformation rate.
- The material is stable: we have to spend mechanical work in order to deform it.

► 1D case

The above observations easily fit into our experience of the 1D load/displacement curve of a steel sample under tension.

In the 1D case it is obvious that (see Fig. 5.1):

$$Total\ \triangle L = Elastic\ \triangle L + Plastic\ \triangle L\ . \tag{5.37a}$$

For the case of infinitesimal strains, dividing the above by ${}^{\circ}L$, we have

$${}^{t}\varepsilon \;=\; {}^{t}\varepsilon^{E} \;+\; {}^{t}\varepsilon^{P} \tag{5.37b}$$

where ${}^{t}\varepsilon$: total axial strain in the sample, ${}^{t}\varepsilon^{E}$: elastic axial strain in the sample and ${}^{t}\varepsilon^{P}$: plastic axial strain in the sample.

During the plastic loading we can also write, following Fig. 5.2, that

$$\triangle\varepsilon = \triangle\varepsilon^{E} + \triangle\varepsilon^{P}\ . \tag{5.37c}$$

Dividing Eq. (5.37c) by $\triangle t$ and taking the limit for $\triangle t \rightarrow 0$ we get for the simple 1D case,

$${}^{t}\dot{\varepsilon} \;=\; {}^{t}\dot{\varepsilon}^{E} \;+\; {}^{t}\dot{\varepsilon}^{P}\ . \tag{5.37d}$$

For the 3D case we generalize the above equation using Eq. (5.24) and get the additive decomposition of the strain-rate tensor,

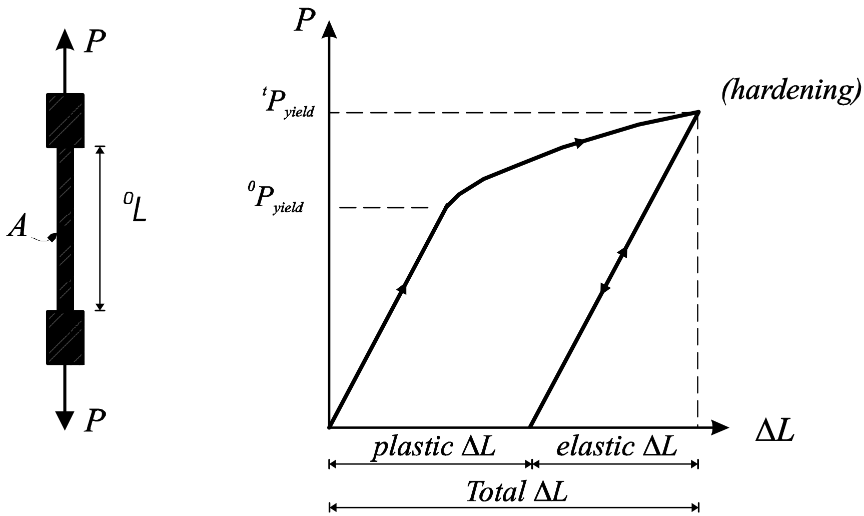


Fig. 5.1. Tensile test of a steel sample

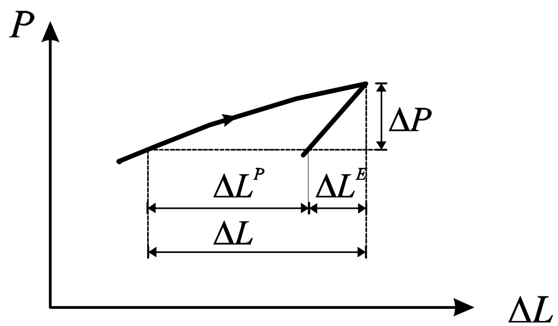


Fig. 5.2. Plastic loading (zoom)

$$ {}^t\underline{\underline{\mathbf{d}}} = {}^t\underline{\underline{\mathbf{d}}}^E + {}^t\underline{\underline{\mathbf{d}}}^P \ . \tag{5.38}$$

It is important to note that, for the moment, we are postulating the above decomposition only for the case of infinitesimal strains, a very important one for standard engineering applications.

In the 1D mechanical test described in Fig. 5.1 it is quite obvious that the *loading condition* is,

$$ {}^t\dot{P} > 0 \ , \tag{5.39a}$$

and when loading:

if ${}^tP < {}^\circ P_{yield}$

$$\triangle L^E = \frac{\triangle P}{AE} \, {}^{\circ}L \tag{5.39b}$$

$$\triangle L^P = 0 \ , \tag{5.39c}$$

if ${}^tP \geqslant {}^{\circ}P_{yield}$

$$\triangle L^E = \frac{\triangle P}{AE} \, {}^{\circ}L \tag{5.39d}$$

$$\triangle L^P > 0 \ , \tag{5.39e}$$

where A is the sample transversal area, that due to the assumption of infinitesimal strains we consider as being constant, and E is the steel Young's modulus.

As is shown in Fig. 5.1 when we reach a load ${}^tP_{yield}$ and *unload*

$${}^t\dot{P} < 0 \tag{5.40a}$$

$$\triangle L^E = \frac{\triangle P}{AE} \, {}^{\circ}L \tag{5.40b}$$

$$\triangle L^P = 0 \ . \tag{5.40c}$$

Let us assume now that, when unloading, we reach a value $P_{UL} \geqslant 0$ and then start loading again:

$${}^t\dot{P} > 0 \tag{5.41a}$$

the behavior will be elastic (Eqs. (5.39b) and (5.39c)) for

$${}^tP < {}^tP_{yield} \tag{5.41b}$$

and elastoplastic (Eqs. (5.39d) and (5.39e)) for

$${}^tP \geqslant {}^tP_{yield} \ . \tag{5.41c}$$

Let us divide in Fig. 5.3 the $P - values$ by A (assumed constant during the deformation) and the $\triangle L - values$ by ${}^{\circ}L$. We get the diagram shown in Fig. 5.3 (Cauchy stress vs. infinitesimal strain).

For a material having a softening behavior, rather than a hardening behavior, we would have the $\sigma - \varepsilon$ diagram shown in Fig. 5.4 . It is evident in this figure that when we go from point "1" to point "2" :

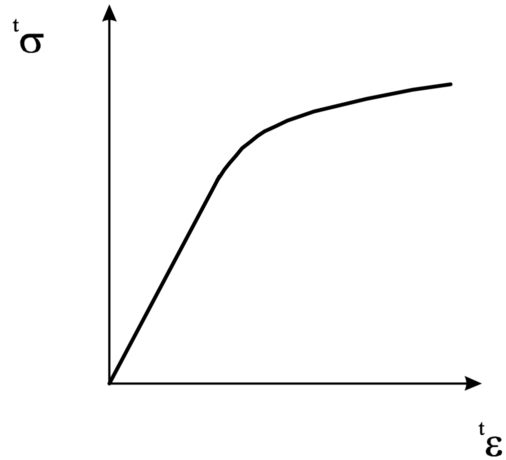


Fig. 5.3. σ-ε for a steel sample

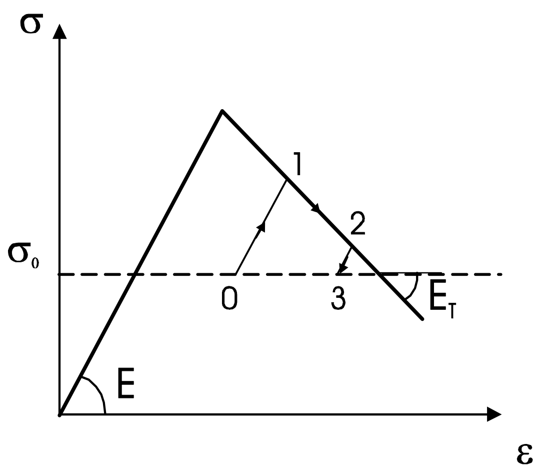


Fig. 5.4. Softening in the $\sigma - \varepsilon$ relation

$$\triangle\sigma \ < \ 0 \tag{5.42a}$$

$$\triangle\varepsilon^P \ > \ 0 \tag{5.42b}$$

and therefore

$$\triangle\sigma \ \triangle\varepsilon^P \ < \ 0 \ . \tag{5.42c}$$

We will now show that the above stress/strain relation is incompatible with the requirement of material stability.

Let us now assume that the 1D sample is in equilibrium at "0" and an *external agent* makes it describe the loading-unloading path $0-1-2-3$, described in Fig. 5.4.

The work performed by the external agent per unit volume of the sample is,

$$\begin{aligned} W_{e.a.} = & \int_0^1 \sigma\, \dot{\varepsilon}\, \mathrm{d}t \ + \int_1^2 \sigma\, \dot{\varepsilon}\, \mathrm{d}t \ + \int_2^3 \sigma\, \dot{\varepsilon}\, \mathrm{d}t \\ & - \int_0^1 \sigma_\circ\, \dot{\varepsilon}\, \mathrm{d}t - \int_1^2 \sigma_\circ\, \dot{\varepsilon}\, \mathrm{d}t - \int_2^3 \sigma_\circ\, \dot{\varepsilon}\, \mathrm{d}t \ . \end{aligned} \tag{5.43}$$

For a stable material it must be

$$W_{e.a.} \geqslant 0 \ . \tag{5.44}$$

Taking into account that in the paths $0-1$ and $2-3$ we are inside the elastic range, we can write

$$\begin{aligned} W_{e.a.} = & \int_0^1 (\sigma - \sigma_\circ)\, \dot{\varepsilon}^E\, \mathrm{d}t \ + \int_1^2 (\sigma - \sigma_\circ)\, \dot{\varepsilon}^E\, \mathrm{d}t \ + \\ & \int_2^3 (\sigma - \sigma_\circ)\, \dot{\varepsilon}^E\, \mathrm{d}t \ + \int_1^2 (\sigma - \sigma_\circ)\, \dot{\varepsilon}^P\, \mathrm{d}t \ \ . \end{aligned} \tag{5.45}$$

The first three integrals on the r.h.s. of the above equation add up to zero because they correspond to a loading-unloading elastic cycle and there is neither energy dissipated nor generated in that cycle.

$$W_{e.a.} = \int_1^2 (\sigma - \sigma_\circ)\, \dot{\varepsilon}^P\, \mathrm{d}t \ \ . \tag{5.46}$$

When "1" and "2" are infinitesimaly close,

$$\mathrm{d}^2 W_{e.a.} = \mathrm{d}\sigma\ \mathrm{d}\varepsilon^P \ \leqslant 0 \ . \tag{5.47}$$

The points on the softening branch are in unstable equilibrium.

$$\boxed{\text{The material shown in Fig. 5.4, with a softening behavior is not stable.}}$$

However, it is possible to obtain diagrams $P - \triangle$ with a softening region; for example:

- In the 1D test of a steel sample after necking (localization). In this case, the hypothesis of constant area is not valid and using at every configuration the actual area for constructing the diagram in Fig. 5.4 (true stress), we get a hardening behavior rather than a softening one.

- In the 1D test of a concrete sample. During the descending branch of the $P - \Delta$ curve there are micro-cracks propagating inside the sample; hence, again the consideration of a uniform stress across the transversal section is not realistic (Ottosen 1986).

In what follows, we will generalize the intuitive concepts of this 1D example to a general 3D formulation.

▶ The general formulation

For the general formulation of an elastoplastic material model we need the following three ingredients:

- A *yield surface* that in the 3D stress space describes the locus of the points where the plastic behavior is initiated.
- A *flow rule* that describes the evolution of the plastic deformations.
- A *hardening law* that describes the evolution of the yield surface during the plastic deformation process.

The yield surface

In the stress space, for a given material we can define a yield surface,

$$ {}^{t}f(\ {}^{t}\underline{\underline{\boldsymbol{\sigma}}}\ ,\ {}^{t}q_i\ i = 1, n\ ) = 0 \tag{5.48}$$

where ${}^{t}\underline{\underline{\boldsymbol{\sigma}}}$ is the Cauchy stress tensor and ${}^{t}q_i$ are internal variables to be defined for every particular yield criterion.

The elastic range is described by the inequality

$$ {}^{t}f < 0 \tag{5.49a}$$

and the plastic range by the equality

$$ {}^{t}f = 0\ . \tag{5.49b}$$

Note that ${}^{t}f > 0$ is not possible in the elastoplastic framework.

Many different yield functions have been proposed over the years to phenomenologically describe the behavior of different materials (Chen 1982, Lubliner 1990). In this section, we concentrate on the yield function that is generally used to describe the behavior of metals: *the von Mises yield function.*

In his experimental work, developed in the 1950s, Bridgman found that for metals, it can be assumed that the yield function is not affected by the confining hydrostatic pressure - at least for not very extreme hydrostatic pressures (Hill 1950 and Johnson & Mellor 1973).

It is important at this point to introduce the following decomposition of the Cauchy stress tensor:

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t}\underline{\underline{\mathbf{s}}} + \frac{1}{3}\left({}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{g}}}\right) {}^{t}\underline{\underline{\mathbf{g}}} \tag{5.50a} $$

which in Cartesian coordinates is written as,

$$ {}^{t}\sigma_{\alpha\beta} = {}^{t}s_{\alpha\beta} + \frac{1}{3}\left({}^{t}\sigma_{\gamma\varphi}\, {}^{t}\delta_{\gamma\varphi}\right) {}^{t}\delta_{\alpha\beta} \ . \tag{5.50b} $$

In the above equation ${}^{t}\underline{\underline{\mathbf{s}}}$ is the *deviatoric Cauchy stress tensor* and $\left[\frac{1}{3}\left({}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{g}}}\right)\right]$ is the *hydrostatic or spherical component of the Cauchy stress tensor*.

Using the above decomposition together with Bridgman experimental observations for the case of metals, we can write Eq. (5.48) as,

$$ {}^{t}f(\, {}^{t}\underline{\underline{\mathbf{s}}}\ ,\ {}^{t}q_i\ \ i = 1, n\ ) = 0 \ . \tag{5.51} $$

Taking into account that the first invariant or trace of the deviatoric Cauchy stress tensor is,

$$ {}^{t}\underline{\underline{\mathbf{s}}} : {}^{t}\underline{\underline{\mathbf{g}}} = 0 \tag{5.52} $$

we can write Eq. (5.51) for an *isotropic material* whose behavior is *symmetric* in the stress space as,

$$ {}^{t}f\left(\, {}^{t}J_2\ ,\ {}^{t}J_3\ ,\ {}^{t}q_i\ \ i = 1, n\ \right) = 0 \tag{5.53} $$

where ${}^{t}J_2$ and ${}^{t}J_3$ are the second and third invariant, respectively, of the deviatoric Cauchy stress tensor.

By isotropic material we mean that the yield condition does not distinguish orientations predefined in the material.

Experimental results indicate that when the yield surface is intersected in the stress space with a plane that forms equal angles with the three principal stress axes, a good approximation for the obtained curve is a circle.

Therefore, von Mises proposed

$$ {}^{t}f\left(\, {}^{t}J_2\ ,\ \ {}^{t}q_i\ \ i = 1, n\ \right) = 0 \ . \tag{5.54} $$

Hence, the von Mises yield function is also known as the ${}^{t}J_2$ *-yield function* (Simo & Hughes 1998).

More specifically the actual form of Eq. (5.54) can be written as,

$$ {}^{t}f = \frac{1}{2}\left(\, {}^{t}\underline{\underline{\mathbf{s}}} - {}^{t}\underline{\underline{\boldsymbol{\alpha}}}\,\right) : \left(\, {}^{t}\underline{\underline{\mathbf{s}}} - {}^{t}\underline{\underline{\boldsymbol{\alpha}}}\,\right) - \frac{\left({}^{t}\sigma_y\right)^2}{3} = 0 \tag{5.55} $$

or,

$$ {}^{t}f = \left[\frac{3}{2}\left(\, {}^{t}\underline{\underline{\mathbf{s}}} - {}^{t}\underline{\underline{\boldsymbol{\alpha}}}\,\right) : \left(\, {}^{t}\underline{\underline{\mathbf{s}}} - {}^{t}\underline{\underline{\boldsymbol{\alpha}}}\,\right)\right]^{1/2} - {}^{t}\sigma_y = 0 \ . \tag{5.56} $$

In the above we have introduced the following internal variables:

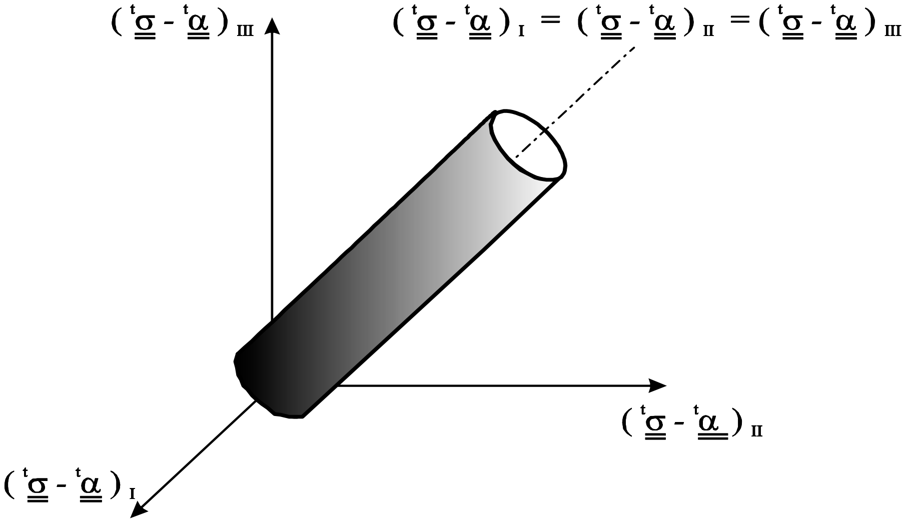


Fig. 5.5. Von Mises yield surface

${}^t\sigma_y$: uniaxial yield stress at the t-configuration; that is to say corresponding to a given plastic deformation. The evolution of ${}^t\sigma_y$ is going to be described by the hardening law.

${}^t\underline{\underline{\alpha}}$: back-stress tensor at the t-configuration. In many metals subjected to cyclic loading it is experimentally observed that the center of the yield surface experiences a motion in the direction of the plastic flow; the back-stress tensor describes this behavior (Mc Clintock & Argon 1966). The evolution of ${}^t\underline{\underline{\alpha}}$ is going to be described by the hardening law. We will show that ${}^t\underline{\underline{\alpha}}$ is a traceless tensor.

In Fig. 5.5 we represent von Mises yield surface in the space of princital (over-)stresses.

The von Mises yield surface in the stress space is a circular cylinder whose axis forms equal angles with each of the coordinates axes.

When, later, we discuss Drucker's definition of stable materials, we will show that for any *stable material* whose behavior is modeled using any yield surface, at every time during the yield surface evolution, this surface has to remain as a *convex surface* in the stress space.

The flow rule

As we stated at the beginning of the section, now we are focusing on the case of infinitesimal strains; hence, the starting point for describing the deformation of an elastoplastic solid during plastic loading is Eq. (5.38) which decomposes the material flow into the addition of an elastic flow plus a plastic flow.

During the material plastic flow there is a *plastic dissipation* per unit volume of mechanical energy given by:

$$ {}^t\mathcal{D} = {}^t\sigma^{ij}\ {}^t d^P_{ij} \tag{5.57}$$

(Hill 1950, Lubliner 1990, Simo & Hughes 1998).

If ${}^t\underline{\underline{\mathbf{d}}}^P = \underline{\underline{\mathbf{0}}}$ (no plastic loading) ${}^t\mathcal{D} = 0$, otherwise the Second Law of Thermodynamics imposes that ${}^t\mathcal{D} > 0$.

For many materials, such as metals, the plastic flow is developed so as to maximize the plastic dissipation.

In mathematical form we can say that for defining the plastic loading we seek for the maximum of ${}^t\mathcal{D}$ under the constraint

$$ {}^t f = 0\ . \tag{5.58a}$$

We define (Luenberger 1984),

$$ {}^t\mathcal{D}^\star = {}^t\mathcal{D}\ -\ {}^t\dot{\lambda}\ {}^t f \tag{5.58b}$$

where ${}^t\dot{\lambda}$ is a Lagrange multiplier used to enforce the constraint in Eq. (5.58a).

The conditions for ${}^t\mathcal{D}$ to attain a maximum under the constraint in Eq. (5.58a) are,

$$ \frac{\partial {}^t\mathcal{D}^\star}{\partial {}^t\sigma^{ij}} = 0 \tag{5.59a}$$

$$ \frac{\partial {}^t\mathcal{D}^\star}{\partial {}^t\dot{\lambda}} = 0\ . \tag{5.59b}$$

The first of the above equations leads to

$$ {}^t d^P_{ij} = {}^t\dot{\lambda}\frac{\partial {}^t f}{\partial {}^t\sigma^{ij}} \tag{5.60}$$

and the second one to Eq. (5.58a).

Note that

$$ \begin{aligned} {}^t f < 0 &\implies {}^t\dot{\lambda} = 0\ (\textit{elastic behavior})\ , \\ {}^t f = 0 &\implies {}^t\dot{\lambda} > 0\ (\textit{plastic loading})\ \ . \end{aligned} $$

The above equations can be summarized using the Kuhn-Tucker conditions for constrained optimization (Luenberger 1984),

$$ {}^t\dot{\lambda}\ {}^t f = 0\ , \tag{5.61a}$$

$$ {}^t\dot{\lambda} \geqslant 0\ , \tag{5.61b}$$

$$ {}^t f \leqslant 0\ . \tag{5.61c}$$

The *Principle of Maximum Plastic Dissipation* states that, for a given plastic strain rate ${}^t\underline{\underline{\mathbf{d}}}^P$, among the possible stresses ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ satisfying the yield

criterion, the plastic dissipation attains its maximum for the actual stress tensor (Simo & Hughes 1998).

The flow described by Eqs. (5.60-5.61c) is called *associative or associated plastic flow* and presents many important properties that we will study in what follows.

A general or nonassociated plastic flow is described by

$$\text{Yield function :} \qquad {}^t f = 0 \tag{5.62a}$$

$$\text{Flow rule :} \qquad {}^t d^P_{ij} = {}^t\dot{\lambda} \frac{\partial^t g}{\partial^t \sigma^{ij}} \tag{5.62b}$$

where ${}^t g$ is called the plastic potential and is different from the yield function.

Example 5.10. ◀◀◀◀◀

For a material model developed using the von Mises yield criterion in Eq. (5.55), we can write in a Cartesian coordinate system:

$$\frac{\partial\ {}^t f}{\partial\ {}^t \sigma_{\alpha\beta}} = \frac{\partial\ {}^t f}{\partial\ {}^t s_{\gamma\delta}} \frac{\partial\ {}^t s_{\gamma\delta}}{\partial\ {}^t \sigma_{\alpha\beta}} = \left({}^t s_{\alpha\beta} - {}^t \alpha_{\alpha\beta} \right) \quad .$$

Hence, in the case of associated plastic flow

$${}^t d^P_{\alpha\beta} = {}^t\dot{\lambda} \left({}^t s_{\alpha\beta} - {}^t \alpha_{\alpha\beta} \right) \quad .$$

Since ${}^t\underline{\underline{\mathbf{s}}}$ and ${}^t\underline{\underline{\boldsymbol{\alpha}}}$ are traceless tensors (see Eqs. (5.78) and (5.52)), it is obvious that,

$${}^t d^P_{\alpha\alpha} = 0 \ ,$$

which is the condition of incompressible plastic flow (see Example 4.4).

The above is, of course, a direct consequence of the fact that due to Bridgman experimental observations the yield function does not include the trace of ${}^t\underline{\underline{\boldsymbol{\sigma}}}$. ◀◀◀◀◀

Stable materials - Drucker's postulate

The principle of maximum plastic dissipation (associated plasticity) implies that the plastic flow develops in the direction of the yield surface external normal, Eq. (5.60). For an arbitrary yield function we schematize this normality rule in Fig.5.6. Note that in the case of infinitesimal strains, it was shown in Eq. (5.24) that ${}^t\underline{\underline{\mathbf{d}}} = {}^t\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}$; which is the nomenclature used in Fig. 5.6.

Let us assume an elasticplastic material that is initially at a stress state ${}^t\underline{\underline{\boldsymbol{\sigma}}}$, inside the elastic range. Let us also assume that an "*external agent*" (one that is independent of whatever has produced the current loads) slowly

applies an incremental load resulting in a stress/strain increment and then slowly removes it. For a *stable material*, the work performed by the external agent in the course of the cycle consisting of the application and removal of the external load is non-negative (Lubliner 1990).

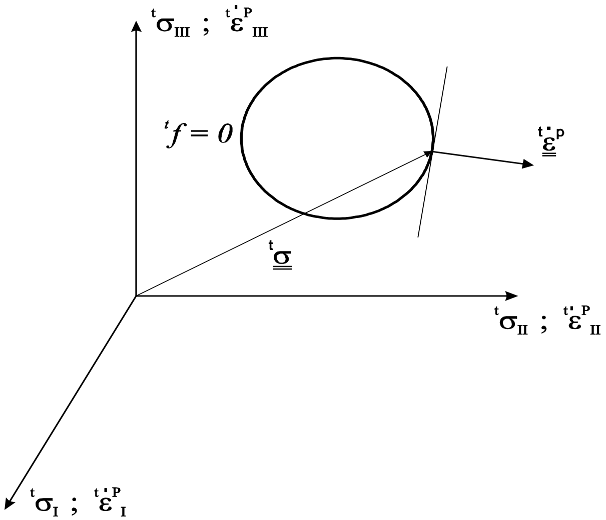


Fig. 5.6. Normality rule in associated plasticity (maximum plastic dissipation)

The above is Drucker's definition of a stable material, also known as *Drucker's postulate.*

In order to analyze the consequences of the above definition let us schematize the above defined cycle in Fig. 5.7.

The total work per unit volume performed during the cycle 0-1-2-0 is,

$$W_{TOTAL} = \int_0^1 \sigma^{ij} \, \mathrm{d}\varepsilon_{ij} + \int_1^2 \sigma^{ij} \, \mathrm{d}\varepsilon_{ij} + \int_2^0 \sigma^{ij} \, \mathrm{d}\varepsilon_{ij} \tag{5.63a}$$

hence, the work performed by the external agent is,

$$\begin{aligned} W_{e.a} = & \int_0^1 \left(\sigma^{ij} - {}^t\sigma^{ij}\right) \, \mathrm{d}\varepsilon_{ij} + \int_1^2 \left(\sigma^{ij} - {}^t\sigma^{ij}\right) \, \mathrm{d}\varepsilon_{ij} \\ & + \int_2^0 \left(\sigma^{ij} - {}^t\sigma^{ij}\right) \, \mathrm{d}\varepsilon_{ij} \, . \end{aligned} \tag{5.63b}$$

For the small strains case, using Eq. (5.38) we obtain,

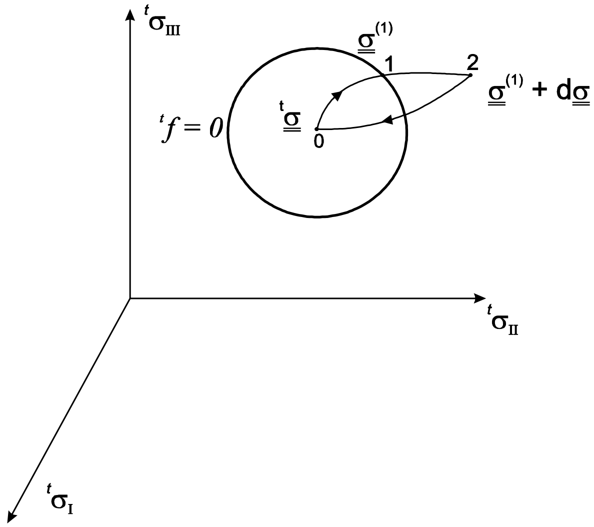


Fig. 5.7. Drucker's postulate (work-hardening material)

Trajectory	$\mathrm{d}\varepsilon_{ij}$
0 - 1	$\mathrm{d}\varepsilon_{ij}^E$
1 - 2	$\mathrm{d}\varepsilon_{ij}^E + \mathrm{d}\varepsilon_{ij}^P$
2 - 0	$\mathrm{d}\varepsilon_{ij}^E$

Taking into account that

$$\oint \sigma^{ij}\ \mathrm{d}\varepsilon_{ij}^E\ = 0 \tag{5.63c}$$

$$\oint {}^t\sigma^{ij}\ \mathrm{d}\varepsilon_{ij}^E\ = 0 \tag{5.63d}$$

we get,

$$W_{e.a} = \int_1^2 \left(\sigma^{ij} -\ {}^t\sigma^{ij}\right)\ \mathrm{d}\varepsilon_{ij}^P\ . \tag{5.63e}$$

For a stable material

$$W_{e.a} \geq 0\ . \tag{5.63f}$$

The above requirement implies that the integrand in Eq. (5.63e) has to be non-negative, that is to say

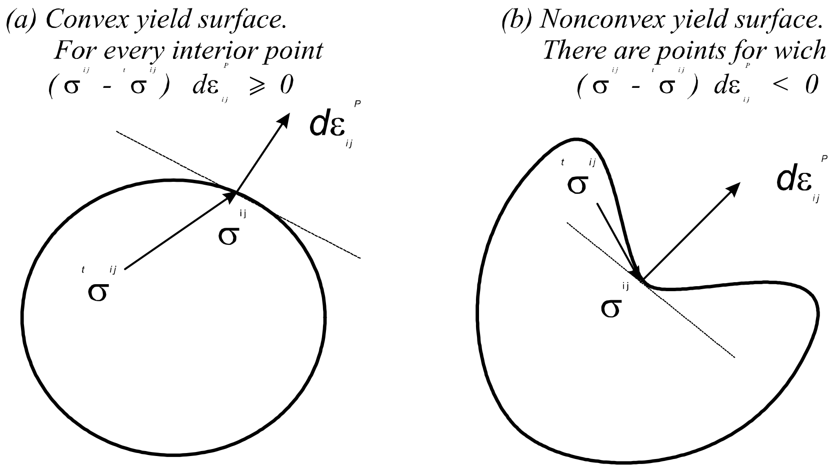


Fig. 5.8. Convexity of the yield surface as a consequence of Drucker's postulate

$$\left(\sigma^{ij} - {}^t\sigma^{ij}\right)\ \mathrm{d}\varepsilon^P_{ij}\ \geq\ 0\ . \tag{5.63g}$$

We now specialize the above equation for the case in which point "2"is inside an infinitesimal neighborhood of "1"; hence,

$$\mathrm{d}\sigma^{ij}\ \mathrm{d}\varepsilon^P_{ij}\ \ \geq\ 0\ . \tag{5.63h}$$

The above constrain has already been derived for the 1D case, see Eq. (5.47).

Equations (5.63g) and (5.63h) are two mathematical constraints that a stable material has to fulfill.

We can rewrite Eq. (5.63h) as,

$$^t\dot{\sigma}^{ij}\ {}^td^P_{ij}\ \geq 0\ . \tag{5.64}$$

The above equation indicates that the plastic strain rate cannot oppose the stress rate (Lubliner 1990).

Note that:

- While ${}^t\dot{\sigma}^{ij}\ {}^td^P_{ij}\ > 0$ indicates a *work-hardening* material, ${}^t\dot{\sigma}^{ij}\ {}^td^P_{ij}\ = 0$ indicates a *perfectly plastic* material.
- Drucker's postulate excludes from the range of stable materials the possibility of strain-softening materials. However, Drucker's postulate has been obtained in the environment of *stress - space plasticity*, i.e. in our plasticity theory the stress is the independent variable. Since the 1960s *strain - space plasticity* formulations have been proposed even though their application has not been widespread yet (Lubliner 1990).

As a consequence of Drucker's postulate, we can show that for a stable material the yield surface has to be a *convex* surface in the stress space.

It is obvious from the two cases schematized in Fig. 5.8 that the nonconvex one fails to fulfill Eq. (5.63g). Therefore, in the environment of stress-space plasticity, a stable material has to have a convex yield surface in the stress space.

Example 5.11. ◀◀◀◀◀

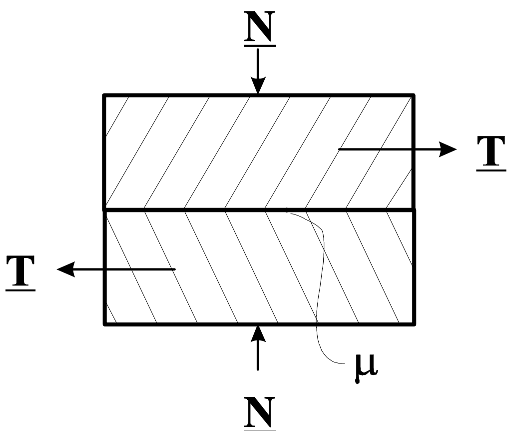
In this example we are going to show that a frictional material cannot be modeled using an associated plasticity formulation (Bažant 1979).

Let us assume the simplest frictional material represented in the following figure: two rigid plates slide one on top of the other, and the sliding surface has a Coulomb friction coefficient μ.

We formulate a yield function in the stress space using T and N (the modulus of $\underline{\mathbf{T}}$ and $\underline{\mathbf{N}}$) as independent variables,

$$^t f = T - \mu N \ ,$$

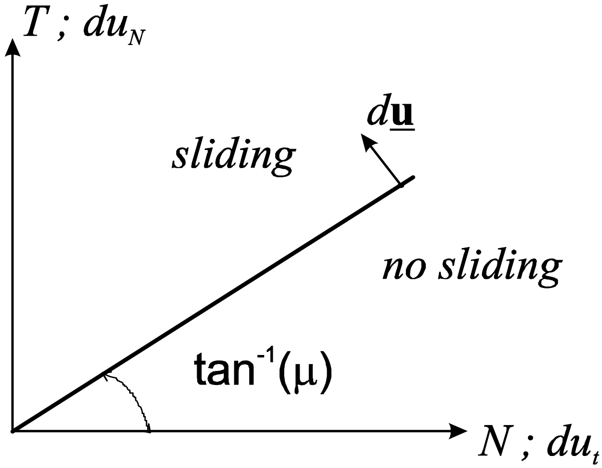
when $^t f < 0 \quad \rightarrow \quad$ *no sliding* and when $^t f = 0 \quad \rightarrow \quad$ *sliding.*



The simplest frictional material

- ∘ The sliding is a nonreversible (plastic) deformation
- ∘ The elastic deformations are zero (rigid-plates hypothesis).

In the following figure we draw the yield surface and the differential plastic displacement obtained under the assumption of associated plasticity.



Yield function and plastic deformation predicted using an associated plasticity formulation (simplest frictional material)

The hypothesis of associated plasticity produces a nonphysical plastic displacement component in the $\underline{\mathbf{N}}$-direction (remember that the plates were assumed to be rigid).
Therefore, to model frictional materials, it is necessary to use nonassociated plasticity formulations (Bažant 1979, Vermeer & de Borst 1984, Dvorkin, Cuitiño & Gioia 1989). ◄◄◄◄◄

Stress - strain relation

Let us consider an elastoplastic material during the process of plastic loading, to relate the stress and strain increments. For the small-strains case in a Cartesian coordinate system Eq. (5.38) can be written as

$$\mathrm{d}\varepsilon_{\alpha\beta} = \mathrm{d}\varepsilon^{E}_{\alpha\beta} + \mathrm{d}\varepsilon^{P}_{\alpha\beta} \ , \tag{5.65a}$$

therefore, for a linear elastic behavior,

$$\mathrm{d}\sigma_{\alpha\beta} = \ C^{E}_{\alpha\beta\gamma\delta} \ \mathrm{d}\varepsilon^{E}_{\gamma\delta} \ . \tag{5.65b}$$

The fourth-order tensor $\underline{\underline{\underline{\underline{\mathbf{C}}}}}^{E}$ is the material elastic constitutive tensor, it is constant for linear elastic behavior and it is a function of the total elastic strains for nonlinear elastic behavior. In this book we are not going to discuss the influence of the plastic deformations on the material elastic properties (damage theory (Lamaitre & Chaboche 1990)).

From Eq. (5.60), for an associated plasticity formulation,

$$\mathrm{d}\varepsilon^{P}_{\alpha\beta} = \mathrm{d}\lambda \frac{\partial \ {}^{t}f}{\partial \ {}^{t}\sigma_{\alpha\beta}} = \mathrm{d}\lambda \frac{\partial \ {}^{t}f}{\partial \ {}^{t}s_{\gamma\delta}} \frac{\partial \ {}^{t}s_{\gamma\delta}}{\partial \ {}^{t}\sigma_{\alpha\beta}} \tag{5.66a}$$

and using Eq. (5.55), the von Mises yield criterion,

$$\frac{\partial\ {}^t f}{\partial\ {}^t s_{\gamma\delta}} = \left(\ {}^t s_{\gamma\delta} -\ {}^t \alpha_{\gamma\delta}\right)\ , \tag{5.66b}$$

also, from Eq. (5.50b),

$$\frac{\partial\ {}^t s_{\gamma\delta}}{\partial\ {}^t \sigma_{\alpha\beta}} = \delta_{\gamma\alpha}\delta_{\delta\beta} - \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta}\ . \tag{5.66c}$$

Taking into account that ${}^t\underline{\underline{\mathbf{s}}}$ and ${}^t\underline{\underline{\boldsymbol{\alpha}}}$ are traceless tensors,

$$\mathrm{d}\varepsilon^P_{\alpha\beta} = \mathrm{d}\lambda \left(\ {}^t s_{\alpha\beta} -\ {}^t \alpha_{\alpha\beta}\right)\ . \tag{5.67a}$$

Using the above together with Eq. (5.65a) in Eq. (5.65b) we get

$$\mathrm{d}\sigma_{\alpha\beta} = \ C^E_{\alpha\beta\gamma\delta} \left[\mathrm{d}\varepsilon_{\gamma\delta} - \mathrm{d}\lambda \left(\ {}^t s_{\gamma\delta} -\ {}^t \alpha_{\gamma\delta}\right)\right]\ . \tag{5.67b}$$

During plastic loading $\mathrm{d}\lambda > 0$; hence, using Eq. (5.61a), we get the consistency condition

$$\mathrm{d}f = 0 \tag{5.68a}$$

which leads to,

$$\frac{\partial^t f}{\partial^t \sigma_{\alpha\beta}}\mathrm{d}\sigma_{\alpha\beta} + \frac{\partial^t f}{\partial^t \varepsilon^P_{\alpha\beta}}\mathrm{d}\varepsilon^P_{\alpha\beta} = 0\ . \tag{5.68b}$$

Using the above, we get

$$\begin{aligned}\left(\ {}^t s_{\alpha\beta} -\ {}^t \alpha_{\alpha\beta}\right)\ C^E_{\alpha\beta\gamma\delta}\ &\left[\mathrm{d}\varepsilon_{\gamma\delta} -\ \mathrm{d}\lambda \left(\ {}^t s_{\gamma\delta} -\ {}^t \alpha_{\gamma\delta}\right)\right] \\ &+ \frac{\partial^t f}{\partial^t \varepsilon^P_{\alpha\beta}}\mathrm{d}\lambda \left(\ {}^t s_{\alpha\beta} -\ {}^t \alpha_{\alpha\beta}\right) = 0\ .\end{aligned} \tag{5.68c}$$

Hence,

$$\mathrm{d}\lambda = \frac{\left(\ {}^t s_{\alpha\beta} -\ {}^t \alpha_{\alpha\beta}\right)\ C^E_{\alpha\beta\gamma\delta}\ \mathrm{d}\varepsilon_{\gamma\delta}}{\left(\ {}^t s_{\varepsilon\zeta} -\ {}^t \alpha_{\varepsilon\zeta}\right)\ C^E_{\varepsilon\zeta\eta\vartheta} \left(\ {}^t s_{\eta\vartheta} -\ {}^t \alpha_{\eta\vartheta}\right) - \frac{\partial^t f}{\partial^t \varepsilon^P_{\varepsilon\zeta}} \left(\ {}^t s_{\varepsilon\zeta} -\ {}^t \alpha_{\varepsilon\zeta}\right)}\ . \tag{5.68d}$$

Replacing in Eq. (5.67b) $d\lambda$ with the above-derived value, we get

$$\begin{aligned}\mathrm{d}\sigma_{\alpha\beta} = \Bigg[C^E_{\alpha\beta\gamma\delta} - \\ \frac{\left(\ {}^t s_{\nu\mu} -\ {}^t \alpha_{\nu\mu}\right)\ C^E_{\alpha\beta\nu\mu}\ C^E_{\varphi\xi\gamma\delta} \left(\ {}^t s_{\varphi\xi} -\ {}^t \alpha_{\varphi\xi}\right)}{\left(\ {}^t s_{\rho\pi} -\ {}^t \alpha_{\rho\pi}\right)\ C^E_{\rho\pi\eta\tau} \left(\ {}^t s_{\eta\tau} -\ {}^t \alpha_{\eta\tau}\right) - \frac{\partial^t f}{\partial^t \varepsilon^P_{\rho\pi}} \left(\ {}^t s_{\rho\pi} -\ {}^t \alpha_{\rho\pi}\right)}\Bigg]\ \mathrm{d}\varepsilon_{\gamma\delta}\ .\end{aligned} \tag{5.69}$$

The term between brackets, ${}^t C^{EP}_{\alpha\beta\gamma\delta}$, represents the Cartesian components of the *continuum tangential elastoplastic constitutive tensor*. Therefore,

$$\mathrm{d}\underline{\underline{\boldsymbol{\sigma}}}\ =\ {}^t\underline{\underline{\underline{\underline{\mathbf{C}}}}}^{EP}\ :\ \mathrm{d}\underline{\underline{\boldsymbol{\varepsilon}}}\ .$$

It is important to note that the following symmetries are present:

- ${}^{t}C^{EP}_{\alpha\beta\gamma\delta} = {}^{t}C^{EP}_{\beta\alpha\gamma\delta}$
- ${}^{t}C^{EP}_{\alpha\beta\gamma\delta} = {}^{t}C^{EP}_{\alpha\beta\delta\gamma}$
- ${}^{t}C^{EP}_{\alpha\beta\gamma\delta} = {}^{t}C^{EP}_{\gamma\delta\alpha\beta}$

It is important to realize that the last symmetry is lost in the case of non-associated plastic models (see Eq. (5.62b)) (Bažant 1979, Vermeer & de Borst 1984).

The hardening law

Following Hill we can say that "*The yield law for a given state of the metal must depend, in some complicated way, on the whole of the previous process of plastic deformation since the last annealing*" (Hill 1950).

In order to solve the equations that describe the elastoplastic deformation process, we have to describe the yield surface evolution during plastic deformation, that is to say, we have to describe the material *hardening.*

The simplest hardening models are :

- Isotropic hardening model.
- Kinematic hardening model.

While the first one does not describe the Bauschinger effect (Hill 1950), the second one was developed to model the basic features of this effect.

In Fig. 5.9 we present a schematic description of the Bauschinger effect. For an initially isotropic hardening material, after loading in tension to ${}^{T}\sigma_y$, when we unload, the yield stress is

- ${}^{T}\sigma_y$ when we reload in tension,
- ${}^{C}\sigma_y$ when we load in compression.

Isotropic hardening

We assume that in Eq. (5.55)

$$ {}^{t}\underline{\underline{\boldsymbol{\alpha}}} = \underline{\underline{\mathbf{0}}} \tag{5.70a}$$

and

$$ {}^{t}\sigma_y = {}^{t}\sigma_y({}^{t}W^P) \ . \tag{5.70b}$$

Hence, the yield surface remains centered and the yield stress is a function of the irreversible work performed on the solid (*work hardening*).

In the above equation ${}^{t}W^P$ is the plastic work per unit volume performed on the material. We can state that,

$$ {}^{t}\mathcal{D} = \frac{D^{t}W^P}{Dt} \quad . \tag{5.70c}$$

The total work per unit volume that has to be spent to deform a solid from its unstrained configuration to a configuration defined by ${}^{t}\varepsilon_{\gamma\delta}$ is, assuming infinitesimal strains,

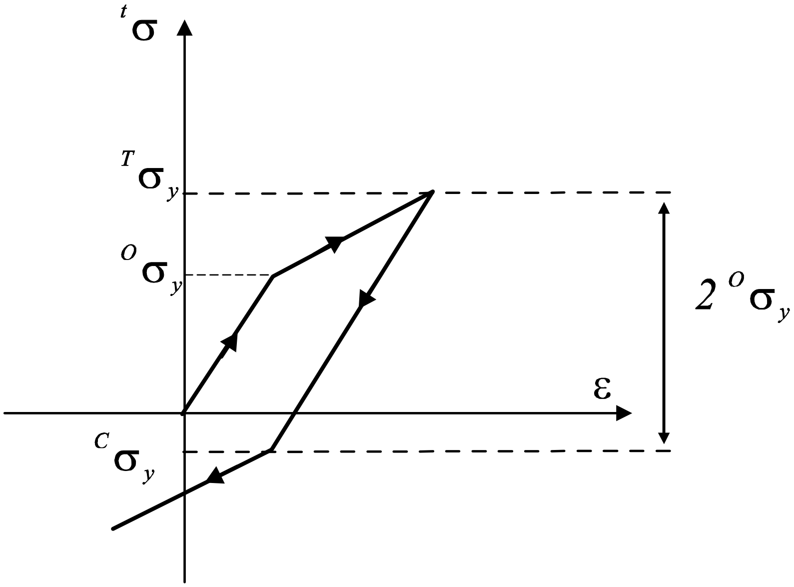


Fig. 5.9. Bauschinger effect

$$ {}^tW = \int_0^{{}^t\varepsilon_{\gamma\delta}} \sigma_{\alpha\beta}\ \mathrm{d}\varepsilon_{\alpha\beta}\ . \tag{5.71a} $$

Using Eq. (5.65a), we get

$$ {}^tW = \int_0^{{}^t\varepsilon^E_{\gamma\delta}} \sigma_{\alpha\beta}\ \mathrm{d}\varepsilon^E_{\alpha\beta}\ + \int_0^{{}^t\varepsilon^P_{\gamma\delta}} \sigma_{\alpha\beta}\ \mathrm{d}\varepsilon^P_{\alpha\beta}\ . \tag{5.71b} $$

Note that we have not yet defined the upper limits of the above integrals.

The first integral on the r.h.s. of Eq. (5.71b) is the elastic (reversible) work per unit volume while the second integral is the plastic (irreversible) work per unit volume.

For an isotropic hardening von Mises material during plastic loading

$$ {}^t\overline{\sigma} = {}^t\sigma_y(\,{}^tW^P)\ , \tag{5.72a} $$

where ${}^t\overline{\sigma}$: equivalent stress $= \sqrt{\frac{3}{2}\ {}^t\underline{\underline{\mathbf{s}}} : {}^t\underline{\underline{\mathbf{s}}}}$.

The rate of plastic work per unit volume is, considering that the plastic flow is incompressible:

$$ {}^t\dot{W}^P = {}^ts_{\alpha\beta}\ {}^td^P_{\alpha\beta}\ . \tag{5.72b} $$

We define, for the isotropic hardening von Mises material the *equivalent plastic strain rate*,

$$ {}^t\overline{d}^P = \sqrt{\frac{2}{3}\ {}^t\underline{\underline{\mathbf{d}}}^P : {}^t\underline{\underline{\mathbf{d}}}^P}\ . \tag{5.73a} $$

Hence,

$$ {}^t\overline{\sigma}\ {}^t\overline{d}^P = \sqrt{\left({}^t\underline{\underline{\mathbf{s}}} : {}^t\underline{\underline{\mathbf{s}}}\right) \left({}^t\underline{\underline{\mathbf{d}}}^P : {}^t\underline{\underline{\mathbf{d}}}^P \right)} \ . \tag{5.73b} $$

Using Eq. (5.60) we can see that ${}^t\underline{\underline{\mathbf{d}}}^P$ and ${}^t\underline{\underline{\mathbf{s}}}$ are collinear tensors when isotropic von Mises plasticity is considered; hence, we can write Eq. (5.73b) as,

$$ {}^t\overline{\sigma}\ {}^t\overline{d}^P = {}^t\underline{\underline{\mathbf{s}}} : {}^t\underline{\underline{\mathbf{d}}}^P = {}^t\dot{W}^P \tag{5.73c} $$

therefore ${}^t\overline{\sigma}$ and ${}^t\overline{d}^P$ are energy conjugate.

We define the equivalent plastic strain as

$$ {}^t\overline{\varepsilon}^P = \int_{\underline{\underline{\mathbf{0}}}}^{{}^t\underline{\underline{\boldsymbol{\varepsilon}}}^P} {}^t\overline{d}^P \ \mathrm{d}t \ . \tag{5.74a} $$

Considering Eqs. (5.72a) and (5.73c)

$$ \mathrm{d}W^P = {}^t\sigma_y({}^tW^P)\ \mathrm{d}\overline{\varepsilon}^P \ . \tag{5.74b} $$

Hence,

$$ \mathrm{d}\sigma_y = \frac{\partial^t\sigma_y}{\partial^tW^P}\ {}^t\sigma_y\ \mathrm{d}\overline{\varepsilon}^P = \frac{\partial^t\sigma_y}{\partial^tW^P}\frac{\partial^tW^P}{\partial^t\overline{\varepsilon}^P}\ \mathrm{d}\overline{\varepsilon}^P = \frac{\partial^t\sigma_y}{\partial^t\overline{\varepsilon}^P}\ \mathrm{d}\overline{\varepsilon}^P \ . \tag{5.74c} $$

Therefore, we can construct a curve,

$$ {}^t\sigma_y = {}^t\sigma_y \left({}^t\overline{\varepsilon}^P\right) \ . \tag{5.75} $$

"*The assumption that one universal stress - strain curve of the form of Eq. (5.75) governs all possible combined - stress loadings of a given material is obviously a very strong one*" (Malvern 1969).

Example 5.12. ◀◀◀◀◀

In a uniaxial tensile test, before the necking is localized, with the I-axis the loading direction and the II and III - axes orthogonal ones, we can write

$$ \begin{aligned} {}^t\sigma_I &= \sigma^* \quad , & {}^ts_I &= \frac{2}{3}\sigma^* \ , \\ {}^t\sigma_{II} &= 0 \quad , & {}^ts_{II} &= -\frac{1}{3}\sigma^* \ , \\ {}^t\sigma_{III} &= 0 \quad , & {}^ts_{III} &= -\frac{1}{3}\sigma^* \ . \end{aligned} $$

Hence,

$$ \overline{\sigma} = \sigma^* \ . $$

Also, due to incompressibility

$$
\begin{aligned}
{}^t\varepsilon_I^P &= \varepsilon_P^* \\
{}^t\varepsilon_{II}^P &= -\frac{1}{2}\varepsilon_P^* \\
{}^t\varepsilon_{III}^P &= -\frac{1}{2}\varepsilon_P^* \ .
\end{aligned}
$$

Hence,

$$
{}^t\overline{\varepsilon}^P = \varepsilon_P^* \ .
$$

Therefore, for an isotropic hardening von Mises material, the complete universal stress - strain curve is determined with only an uniaxial test. ◀◀◀◀◀

Kinematic hardening

In this hardening model, which was developed to simulate the basic features of the Bauschinger effect, we assume:

$$
{}^t\sigma_y = {}^\circ\sigma_y = const. \tag{5.76}
$$

Kinematic hardening represents a translation of the yield surface in the stress space by shifting its center. This is in fairly good agreement with the Bauschinger effect for those materials whose stress-strain curve in the work-hardening range can be approximated by a straight line ("linear hardening") (Lubliner 1990).

Prager, in his kinematic hardening model, assumes a linear hardening (Malvern 1969):

$$
{}^t\dot{\alpha}_{ij} = c \ \ {}^t d_{ij}^P \tag{5.77}
$$

where c is a constant.

Since ${}^t d_{\alpha\alpha}^P = 0$ (incompressibility), the back-stress tensor ${}^t\underline{\underline{\boldsymbol{\alpha}}}$ is also traceless; that is to say

$$
{}^t\underline{\underline{\boldsymbol{\alpha}}} \ : \ {}^t\underline{\underline{\mathbf{g}}} \ = \ 0 \quad . \tag{5.78}
$$

In the nine-dimensional stress space, the yield surface is displaced in the direction of its external normal at the load point.

Example 5.13. ◀◀◀◀◀

We consider the 1D case of monotonic loading, also considered in the previous example.

Considering the yield function in Eq. (5.55) under the constraints in Eqs. (5.76) and (5.77); and imposing that during loading

$$
{}^t\dot{f} = 0 \ ,
$$

we get,

$$c = \frac{\dot{\bar{\sigma}}}{\frac{3}{2}\dot{\bar{\varepsilon}}} \; .$$

Therefore, the hardening parameter, c, is constant for linear hardening materials. ◀◀◀◀◀

When using the kinematic hardening model, as also discussed for the isotropic hardening model, the complete universal stress-strain curve is determined with one axial test.

It is important to realize that the results for monotonic loading are exactly coincident when considering either the isotropic or the kinematic hardening models.

5.2.6 Elastoplastic material model under finite strains

▶ Hypoelastic models

Following what has been done for the case of elastoplastic materials under infinitesimal strains, we start this Section by assuming a hypoelastic description of the deformation process.

Referring the problem to the equilibrium configuration at time t, we may again use

$$ {}^t\underline{\underline{\mathbf{d}}} = {}^t\underline{\underline{\mathbf{d}}}^E + {}^t\underline{\underline{\mathbf{d}}}^P \; . \tag{5.79}$$

As we have shown in Example 2.22, ${}^t\underline{\underline{\mathbf{d}}}$ is an objective strain rate; therefore, considering Eq. (5.79), both ${}^t\underline{\underline{\mathbf{d}}}^E$ and ${}^t\underline{\underline{\mathbf{d}}}^P$ have to be considered as objective strain rates too. Hence, we can extend the constitutive laws used for infinitesimal strains and write

$$ {}^t\overset{\circ}{\underline{\underline{\boldsymbol{\sigma}}}} = {}^t\underline{\underline{\underline{\underline{\mathbf{c}}}}}^E : {}^t\underline{\underline{\mathbf{d}}}^E \tag{5.80a}$$

$$ {}^t\underline{\underline{\mathbf{d}}}^P = \dot{\lambda}\, {}^t\underline{\underline{\mathbf{s}}} \; . \tag{5.80b}$$

In the above equation ${}^t\overset{\circ}{\underline{\underline{\boldsymbol{\sigma}}}}$ is an objective stress rate such as the Oldroyd stress rate (see Sect. 3.4) and ${}^t\underline{\underline{\underline{\underline{\mathbf{c}}}}}^E$ is the spatial elasticity tensor.

Equation (5.80a) is the hypoelastic expression of an elastic behavior and we expect it to conserve energy; that is to say, it has to be the hypoelastic expression of a hyperelastic material behavior.

In the next example we show that only for the case of infinitesimal strains, it is valid to assume that ${}^{t}\underline{\underline{\underline{\underline{\mathbf{c}}}}}^{E}$ is a *constant and isotropic* tensor.

Example 5.14. ◀◀◀◀◀

Following (Simo & Pister 1984) we will show that the hypoelastic equation

$$ {}^{t}\overset{\circ}{\sigma}{}^{ij} = {}^{t}c^{ijkl}\;{}^{t}d_{kl} $$

fails to model a hyperelastic behavior when the fourth-order constitutive tensor is constant and isotropic (see also Example 5.6).

(a) Calculation of ${}^{t}\dot{\sigma}^{ij}$

Using Eq. (3.35), we get

$$ {}^{t}\overset{\circ}{\underline{\underline{\boldsymbol{\sigma}}}} = {}^{t}\dot{\underline{\underline{\boldsymbol{\sigma}}}} - {}^{t}\underline{\underline{\mathbf{l}}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} - {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t}\underline{\underline{\mathbf{l}}}^{T} . $$

Also, using the definition of the second Piola-Kirchhoff stress tensor, we can write

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t}J^{-1}\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{S}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} , \tag{A} $$

where ${}^{t}J = \frac{{}^{\circ}\rho}{{}^{t}\rho}$.

Using the result in Example 5.3, we can write

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{S}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{S}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{C}}}\right) . $$

From Eq. (A)

$$ \begin{aligned} {}^{t}\dot{\underline{\underline{\boldsymbol{\sigma}}}} = & -\frac{{}^{t}\dot{J}}{{}^{t}J^{2}}\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{S}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{C}}}\right) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} + {}^{t}J^{-1}\; {}^{t}\underline{\underline{\mathbf{l}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{S}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{C}}}\right) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \\ & + {}^{t}J^{-1}\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot \frac{\partial {}^{t}_{\circ}\underline{\underline{\mathbf{S}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{C}}}\right)}{\partial {}^{t}_{\circ}\underline{\underline{\mathbf{C}}}} \cdot {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{C}}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \\ & + {}^{t}J^{-1}\; {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{S}}}\left({}^{t}_{\circ}\underline{\underline{\mathbf{C}}}\right) \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \cdot {}^{t}\underline{\underline{\mathbf{l}}}^{T} . \end{aligned} $$

Also,

$$ {}^{t}c^{kmpq} = 2\; J^{-1}\; {}^{t}_{\circ}X^{k}{}_{K}\; {}^{t}_{\circ}X^{m}{}_{M}\; {}^{t}_{\circ}X^{p}{}_{P}\; {}^{t}_{\circ}X^{q}{}_{Q}\; \frac{\partial\, {}^{t}_{\circ}S^{KM}}{\partial\, {}^{t}_{\circ}C_{PQ}} . \tag{5.81} $$

Using Example 4.1 we can write ${}^{t}\dot{J} = {}^{t}J\;(\underline{\boldsymbol{\nabla}} \cdot {}^{t}\underline{\mathbf{v}}) = {}^{t}J\left({}^{t}\underline{\underline{\mathbf{d}}} : {}^{t}\underline{\underline{\mathbf{g}}}\right)$ and taking into account that ${}^{t}g^{ab}|_{i} = 0$, after same algebra, we get, using components, the equation (Truesdell & Noll 1965)

$$ {}^{t}\dot{\sigma}^{km} = {}^{t}h^{kmpq}\ {}^{t}l_{pq} \tag{B.1} $$

$$ {}^{t}h^{kmpq} = -\ {}^{t}\sigma^{km}\ {}^{t}g^{pq} + {}^{t}g^{kp}\ {}^{t}\sigma^{qm} + {}^{t}c^{kmpq} + {}^{t}\sigma^{kq}\ {}^{t}g^{mp}\ . \tag{B.2} $$

In the derivation of the above equation we made use of the symmetry condition ${}^{t}c^{kmpq} = {}^{t}c^{kmqp}$.

(b) The Bernstein formula

Following (Truesdell & Noll 1965) we now present the conditions that ${}^{t}h^{kmpq}$ in Eqs. (B.1-B.2) needs to fulfill in order to model a hyperelastic material behavior.

For a hyperelastic material behavior, we can write, using Eq. (5.4e),

$$ {}^{t}\sigma^{ij} = 2\ {}^{t}\rho \frac{\partial\, {}^{t}\mathtt{u}}{\partial\, {}^{t}g_{ij}}\ , $$

where ${}^{t}\mathtt{u}$ is the elastic energy per unit mass.

We can also write the following functional dependence,

$$ {}^{t}\sigma^{km} = \varphi^{km}\left({}^{t}_{\circ}X^{a}_{A}\right)\ . \tag{C} $$

Hence,

$$ {}^{t}\dot{\sigma}^{km} = \frac{\partial \varphi^{km}}{\partial\, {}^{t}_{\circ}X^{a}_{A}}\ {}^{t}l^{a}_{\ b}\ {}^{t}_{\circ}X^{b}_{A}\ . $$

Therefore, using the above and Eq. (B.1), we get

$$ \frac{\partial \varphi^{km}}{\partial\, {}^{t}_{\circ}X^{a}_{A}}\ {}^{t}l^{a}_{\ b}\ {}^{t}_{\circ}X^{b}_{A} = {}^{t}h^{kmpq}\ {}^{t}l_{pq}\ , $$

and after some algebra

$$ \frac{\partial \varphi^{km}}{\partial\, {}^{t}_{\circ}X^{p}_{A}} = {}^{t}h^{km}{}_{p}{}^{q}\ \left({}^{t}_{\circ}X^{-1}\right)^{A}_{\ q}\ . \tag{D} $$

The above is written in (Truesdell & Noll 1965) as Eq. (100.24). The function φ^{km} has to fulfill the following condition to be a potential function,

$$ \frac{\partial^{2} \varphi^{km}}{\partial\, {}^{t}_{\circ}X^{p}_{A}\ \partial\, {}^{t}_{\circ}X^{r}_{B}} = \frac{\partial^{2} \varphi^{km}}{\partial\, {}^{t}_{\circ}X^{r}_{B}\ \partial\, {}^{t}_{\circ}X^{p}_{A}}\ . \tag{E} $$

Using in Eq. (E) the equality (D) and considering that (Truesdell & Toupin 1960) the equation

$$\left({}^t_\circ X^{-1}\right)^B_{\ q}\ {}^t_\circ X^q_{\ M} \ = \ \delta^B_M \tag{5.82}$$

leads to

$$\frac{\partial\ \left({}^t_\circ X^{-1}\right)^B_{\ q}}{\partial\ {}^t_\circ X^p_{\ A}} \ = \ - \ \left({}^t_\circ X^{-1}\right)^B_{\ p}\ \left({}^t_\circ X^{-1}\right)^A_{\ q} \tag{5.83}$$

we finally get the Bernstein formula:

$$\frac{\partial^t h^{kmpq}}{\partial^t \sigma^{rs}}\ {}^t h^{rsjl} - \frac{\partial^t h^{kmjl}}{\partial^t \sigma^{rs}}\ {}^t h^{rspq} - \ {}^t h^{kmpl}\ {}^t g^{jq} + \ {}^t h^{kmjq}\ {}^t g^{pl} \ = 0\ . \tag{F}$$

(c) The constant and isotropic constitutive tensor

Finally, we are going to show that when using a constant and isotropic constitutive tensor in Eq. (5.80a), the Bernstein formula is not fulfilled and therefore, a hyperelastic material behavior cannot be modeled.
A general constant isotropic tensor is written as (Aris 1962)

$${}^t c^{kmjl} = \lambda\ {}^t g^{km}\ {}^t g^{jl} \ + \ \mu\ \left({}^t g^{kj}\ {}^t g^{ml} \ + \ {}^t g^{kl}\ {}^t g^{mj}\right)$$

then we replace in Eq. (F) and we find that the equality can be satisfied only for $(\lambda + \mu) \ = \ 0$. Since this constraint on the material properties does not correspond to a physical acceptable material model, we conclude that it is not possible to use a constant and isotropic constitutive tensor in Eq. (5.80a) to model a hyperelastic material behavior.________________◀◀◀◀◀

▶ The multiplicative decomposition of the deformation gradient

The use of hypoelastic models to numerically analyze finite strains elastoplastic problems leads to many difficulties:

- The strain- and stress-rate measures have to be both objective and incrementally objective (i.e. $\Delta\underline{\underline{\varepsilon}}$ and $\Delta\underline{\underline{\sigma}}$ have to be objective tensors). As we have shown in Sect. 3.4 the Jaumann stress-rate, which is an objective stress-rate measure, is not suitable for producing an objective incremental stress measure.
- The limitation discussed in Example 5.14 to represent an elastic behavior using a hypoelastic constitutive model.

An alternative kinematic formulation to numerically analyze finite-strain elastoplastic problems can be found in (Lee & Liu 1967, Lee 1969). Lee's kinematic formulation: *the multiplicative decomposition of the deformation gradient*, is based on a micromechanical model of single-crystal metal plasticity (Simo & Hughes 1998).

Lee's multiplicative decomposition of the deformation gradient has been the basis of the hyperelastic model for finite-strain elastoplasticity developed by Simo and Ortiz (Simo & Ortiz 1985, Simo 1988, Simo & Hughes 1998).

We will now present an "intuitive description" of Lee's multiplicative decomposition of the deformation gradient (Lubliner 1990).

Let us consider a reference configuration that we assume to be unstressed and unstrained and the spatial configuration corresponding to a certain time "t". In this spatial configuration, we have reversible (elastic) deformations and permanent (plastic) deformations.

If we now cut the spatial configuration into hexahedric infinitesimal volumes and if we assume for each of those volumes an *elastic unloading process,* we obtain a *stress-free configuration* that is called the *intermediate configuration.* Obviously, the infinitesimal hexahedrals will not match together in the stress free or intermediate configuration to form a continuum because compatibility was lost when we divided the spatial continuum into infinitesimal parts.

Therefore, the intermediate configuration is not a proper configuration because there is not a bijective mapping between the material particles and $\Re^3$.

In Fig. 5.10 we schematize the three configurations: the reference one, the intermediate one and the spatial one. We also indicate in this figure the arbitrary coordinates defined on the reference and spatial configurations and the corresponding deformation gradients.

From this figure

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{E} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{P} \,. \tag{5.84} $$

<u>Notes</u>:

- Since the intermediate configuration, due to its lack of compatibility is not a proper configuration, the tensors ${}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{E}$ and ${}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{P}$ cannot be calculated using the definition of a deformation gradient tensor (Eq. (2.23)).
- The mapping represented by ${}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{E}$ is purely elastic and the stresses in the spatial configuration are developed during this mapping.

The velocity gradient is defined in the spatial configuration and the following relation holds

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}_{\circ}^{t}\underline{\underline{\dot{\mathbf{X}}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{-1} \,. \tag{5.85a} $$

Using the multiplicative decomposition in Eq. (5.84), we obtain

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}_{\circ}^{t}\underline{\underline{\dot{\mathbf{X}}}}^{E} \cdot \left({}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{E}\right)^{-1} + {}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{E} \cdot {}_{\circ}^{t}\underline{\underline{\dot{\mathbf{X}}}}^{P} \cdot \left({}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{P}\right)^{-1} \cdot \left({}_{\circ}^{t}\underline{\underline{\mathbf{X}}}^{E}\right)^{-1} \,. \tag{5.85b} $$

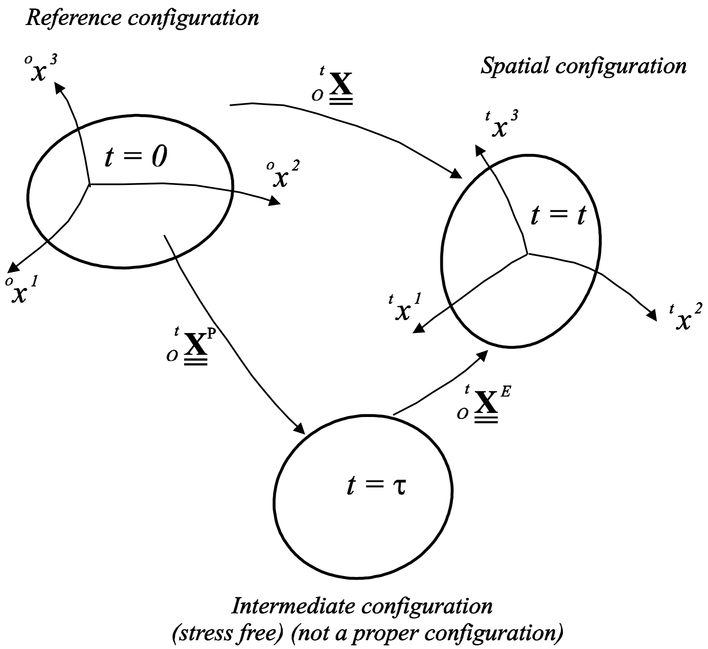


Fig. 5.10. Lee's multiplicative decomposition of the deformation gradient

We call ${}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P} = {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}}^{P} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{P}\right)^{-1}$ a tensor defined in the intermediate configuration, and we rewrite the above equation as:

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}}^{E} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E}\right)^{-1} + {}^{t}_{\circ}X_{e\star}\left({}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P}\right) \ . \tag{5.85c} $$

In the above equation we used the notation ${}^{t}_{\circ}X_{e\star}\,(.)$ to indicate the push-forward of the components of the tensor $(.)$ from the intermediate configuration to the spatial configuration.

At the intermediate configuration, we can make the following additive decomposition

$$ {}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P} = {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^{P} + {}^{t}\underline{\underline{\bar{\boldsymbol{\omega}}}}^{P} \ . \tag{5.85d} $$

A standard hypothesis is that if the material under consideration has isotropic elastic properties, we can impose

$$ {}^{t}\underline{\underline{\bar{\boldsymbol{\omega}}}}^{P} \equiv 0 \ . \tag{5.86} $$

The above hypothesis was used in (Weber & Anand 1990, Eterovic & Bathe 1990, Dvorkin, Pantuso & Repetto 1994).

By doing the polar decomposition of the elastic and plastic deformation gradients, obtained in Eq. (5.84) via Lee's multiplicative decomposition, we get

$$ {}^t_\circ\underline{\underline{\mathbf{X}}}^E = {}^t_\circ\underline{\underline{\mathbf{R}}}^E \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}^E = {}^t_\circ\underline{\underline{\mathbf{V}}}^E \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}^E \tag{5.87a}$$

$$ {}^t_\circ\underline{\underline{\mathbf{X}}}^P = {}^t_\circ\underline{\underline{\mathbf{R}}}^P \cdot {}^t_\circ\underline{\underline{\mathbf{U}}}^P = {}^t_\circ\underline{\underline{\mathbf{V}}}^P \cdot {}^t_\circ\underline{\underline{\mathbf{R}}}^P \ . \tag{5.87b}$$

We can also define the elastic Hencky strain tensor as

$$ {}^t_\circ\underline{\underline{\mathbf{H}}}^E = \ln\left({}^t_\circ\underline{\underline{\mathbf{U}}}^E\right) \ . \tag{5.88}$$

▶ Stresses and the yield criterion

Considering an *elastically isotropic material*, the energy conjugate of ${}^t_\circ\underline{\underline{\mathbf{H}}}^E$ is (see Sect. 3.3.4),

$$ {}^t\underline{\underline{\mathbf{\Gamma}}} = {}^t_\circ R_e^\star({}^t\underline{\underline{\boldsymbol{\tau}}}) \ . \tag{5.89}$$

Following the work by Lee (Lee & Liu 1967, Lee 1969) we formulate the yield criterion in terms of Kirchhoff stresses. Since we are interested in the modeling of elastoplastic deformation processes in metals, we use the von Mises (J_2) yield criteria combined with isotropic/kinematic hardening and we have

$$ {}^tf = \left[\frac{3}{2}\left({}^t\underline{\underline{\boldsymbol{\tau}}}_D - {}^t\underline{\underline{\boldsymbol{\alpha}}}\right) : \left({}^t\underline{\underline{\boldsymbol{\tau}}}_D - {}^t\underline{\underline{\boldsymbol{\alpha}}}\right)\right]^{\frac{1}{2}} - {}^t\sigma_y = 0 \ . \tag{5.90}$$

In the above,
${}^t\underline{\underline{\boldsymbol{\tau}}}_D$: deviatoric Kirchhoff stress tensor;
${}^t\underline{\underline{\boldsymbol{\alpha}}}$: back-stress tensor (traceless);
${}^t\sigma_y$: yield stress in the t-configuration.

Example 5.15. ◀◀◀◀◀
In (Lee 1969), Lee noted that the rate of plastic work invested per unit volume of the reference configuration is,

$$ {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}}^P \left(\frac{{}^\circ\rho}{{}^t\rho}\right) \tag{A}$$

where ${}^\circ\rho$ and ${}^t\rho$ are the densities of the reference and spatial configurations. Since the plastic flow is incompressible, then only during the elastic deformation can the densities change.
The plastic work in Eq. (A) is related to the material plastic hardening. Equation (A) indicates a decrease in hardening when we increase the hydrostatic pressure (${}^t\rho > {}^\circ\rho$) . To avoid this coupling between the elastic and plastic behavior, Lee suggests using, in the finite deformation elastoplastic formulation, the Kirchhoff stress tensor rather than the Cauchy one in the yield criterion. Obviously, for infinitesimal strains, both stress tensors are identical. ◀◀◀◀◀

The tensors defined by

$$ {}^{t}B^{IJ} = {}^{t}_{\circ}R_e^{\star}\left({}^{t}\alpha^{ij}\right) \tag{5.91a}$$

$$ {}^{t}\Gamma_D^{IJ} = {}^{t}_{\circ}R_e^{\star}\left({}^{t}\tau_D^{ij}\right) $$

are traceless.

By doing a ${}^{t}_{\circ}R_e$-pull-back of Eq. (5.90), we get

$$ {}^{t}f = \left[\frac{3}{2}\left({}^{t}\underline{\underline{\boldsymbol{\Gamma}}}_D - {}^{t}\underline{\underline{\mathbf{B}}}\right) : \left({}^{t}\underline{\underline{\boldsymbol{\Gamma}}}_D - {}^{t}\underline{\underline{\mathbf{B}}}\right)\right]^{\frac{1}{2}} - {}^{t}\sigma_y = 0 \,. \tag{5.92}$$

We consider the following evolution equations (Eterovic & Bathe 1990, Dvorkin, Pantuso & Repetto 1994):

$$ {}^{t}\dot{\sigma}_y = \beta\ h\ {}^{t}\bar{d}^P \tag{5.93a}$$

$$ {}^{t}\dot{\underline{\underline{\mathbf{B}}}} = \frac{2}{3}\left(1-\beta\right)\ h\ {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^P \,. \tag{5.93b}$$

In the above, $h = h\left({}^{t}\bar{e}^P\right)$ is the hardening module, ${}^{t}\bar{d}^P$ is defined in (5.73a) and $\beta \in [0,1]$ is the hardening ratio; $\beta = 0$ corresponds to purely kinematic hardening and $\beta = 1$ corresponds to purely isotropic hardening.

We also define the equivalent plastic strain as:

$$ {}^{t}\bar{e}^P = \int_0^t {}^{t}\bar{d}^P\ \mathrm{d}t \,. \tag{5.94}$$

▶ Energy dissipation

We now introduce ${}^{t}\psi$ as the free energy at the spatial configuration per unit volume of the reference configuration.

Considering that the mechanical problem is uncoupled from the thermal problem, we can write (Simo & Hughes 1998)

$$ {}^{t}\underline{\underline{\boldsymbol{\tau}}} : {}^{t}\underline{\underline{\mathbf{d}}} - {}^{t}\dot{\psi} \geqslant 0 \,. \tag{5.95a}$$

The above equation, known as the *Clausius-Duhem* inequality, is a restriction imposed by the Second Law of Thermodynamics (Simo & Hughes 1998):

- For the elastic case the deformation is reversible and the equal sign holds.
- When there are plastic deformations the process is irreversible and the greater-than sign holds.

Therefore, using Lee's multiplicative decomposition, we can write

$$ {}^{t}_{\circ}X^{P}_{\alpha\beta} = \delta_{\alpha\beta} \quad \longrightarrow \quad {}^{t}\underline{\underline{\boldsymbol{\tau}}} : {}^{t}\underline{\underline{\mathbf{d}}} - {}^{t}\dot{\psi} = 0 \tag{5.95b} $$

$$ {}^{t}_{\circ}X^{P}_{\alpha\beta} \neq \delta_{\alpha\beta} \quad \longrightarrow \quad {}^{t}\underline{\underline{\boldsymbol{\tau}}} : {}^{t}\underline{\underline{\mathbf{d}}} - {}^{t}\dot{\psi} > 0 \,. \tag{5.95c} $$

For the free energy of the purely mechanical problem, we can state the following functional relation:

$$ {}^{t}\psi = {}^{t}\psi \left({}^{t}_{\circ}\underline{\underline{\mathbf{H}}}^{E},\ {}^{t}\sigma_{y},\ {}^{t}\underline{\underline{\mathbf{B}}} \right) \,. \tag{5.96a} $$

Following Simo 1988, we use the following uncoupled expression for the free energy,

$$ {}^{t}\psi = {}^{t}\psi_{e} \left({}^{t}_{\circ}\underline{\underline{\mathbf{H}}}^{E} \right) + {}^{t}\psi_{p} \left({}^{t}\sigma_{y},\ {}^{t}\underline{\underline{\mathbf{B}}} \right) \,. \tag{5.96b} $$

Considering a metal, ${}^{t}\psi_{e}$ is the elastic free energy that we identify with the atomic lattice deformation energy and ${}^{t}\psi_{p}$ is the energy associated with atomic lattice defects (e.g. dislocations) (Lubliner 1990).

From Eq. (5.85b),

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}^{t}\underline{\underline{\mathbf{l}}}^{E} + {}^{t}\underline{\underline{\mathbf{l}}}^{P} \tag{5.97a} $$

$$ {}^{t}\underline{\underline{\mathbf{l}}}^{E} = {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{X}}}}^{E} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{-1} = {}^{t}\underline{\underline{\mathbf{d}}}^{E} + {}^{t}\underline{\underline{\boldsymbol{\omega}}}^{E} \tag{5.97b} $$

$$ {}^{t}\underline{\underline{\mathbf{l}}}^{P} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \cdot {}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{-1} = {}^{t}\underline{\underline{\mathbf{d}}}^{P} + {}^{t}\underline{\underline{\boldsymbol{\omega}}}^{P} \,. \tag{5.97c} $$

Therefore,

$$ {}^{t}\underline{\underline{\boldsymbol{\tau}}} : {}^{t}\underline{\underline{\mathbf{d}}} = {}^{t}\underline{\underline{\boldsymbol{\tau}}} : \left({}^{t}\underline{\underline{\mathbf{d}}}^{E} + {}^{t}\underline{\underline{\mathbf{d}}}^{P} \right) \,. \tag{5.98} $$

After some algebra, we can write

$$ {}^{t}\underline{\underline{\boldsymbol{\tau}}} : {}^{t}\underline{\underline{\mathbf{d}}}^{P} = \left[\left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{T} \cdot {}^{t}\underline{\underline{\boldsymbol{\tau}}} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{-T} \right] : {}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P} \,. \tag{5.99} $$

Since we restrict this formulation to the case of isotropic elastic properties, we can write (see Sect. 3.3.4)

$$ {}^{t}\underline{\underline{\boldsymbol{\tau}}} : {}^{t}\underline{\underline{\mathbf{d}}}^{E} = {}^{t}\underline{\underline{\boldsymbol{\Gamma}}} : {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{H}}}}^{E} \tag{5.100} $$

and the Clausius-Duhem inequality takes the form,

$$ \left({}^{t}\underline{\underline{\boldsymbol{\Gamma}}} - \frac{\partial\, {}^{t}\psi_{e}}{\partial\, {}^{t}_{\circ}\underline{\underline{\mathbf{H}}}^{E}} \right) : {}^{t}_{\circ}\underline{\underline{\dot{\mathbf{H}}}}^{E} + \left[\left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{T} \cdot {}^{t}\underline{\underline{\boldsymbol{\tau}}} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{-T} \right] : {}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P} - {}^{t}\dot{\psi}_{p} \geqslant 0 \,. \tag{5.101} $$

Since the above must also be valid for the case of an elastic deformation, we obtain

$$ {}^{t}\underline{\underline{\boldsymbol{\Gamma}}} = \frac{\partial\, {}^{t}\psi_e}{\partial\, {}^{t}_{\circ}\underline{\underline{\mathbf{H}}}^{E}} \; . \tag{5.102} $$

We define as *dissipation* (Simo 1988):

$$ {}^{t}\mathcal{D} = \left[\left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{T} \cdot {}^{t}\underline{\underline{\boldsymbol{\tau}}} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \right)^{-T} \right] : {}^{t}\underline{\underline{\bar{\mathbf{l}}}}^{P} - {}^{t}\dot{\psi}_p \geqslant 0 \; . \tag{5.103} $$

Considering that:

- For elastically isotropic materials the tensors ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$, ${}^{t}_{\circ}\underline{\underline{\mathbf{H}}}^{E}$ and therefore ${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}^{E}$ are *collinear.*
- The contraction of a symmetric tensor with a skew-symmetric one is zero.

We get,

$$ {}^{t}\mathcal{D} = {}^{t}\underline{\underline{\boldsymbol{\Gamma}}} : {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^{P} - {}^{t}\dot{\psi}_p \geqslant 0 \; . \tag{5.104} $$

We search for the value of ${}^{t}\underline{\underline{\boldsymbol{\Gamma}}}$ that maximizes the dissipation under the unilateral constraint ${}^{t}f \leqslant 0$.

For the case ${}^{t}f < 0$ (elastic loading or unloading)

$$ {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^{P} = \underline{\underline{\mathbf{0}}} \; . \tag{5.105} $$

For the case ${}^{t}f = 0$ (plastic loading) to solve the minimization problem, we use the Kuhn-Tucker conditions (Luenberger 1984) and we obtain the well-known associated flow rule (see Eqs. (5.57) to (5.61c) for the case of infinitesimal strains):

$$ {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^{P} = {}^{t}\dot{\lambda}\, \frac{\frac{3}{2}\left({}^{t}\underline{\underline{\boldsymbol{\Gamma}}}_D - {}^{t}\underline{\underline{\mathbf{B}}} \right)}{\sqrt{\frac{3}{2}\left({}^{t}\underline{\underline{\boldsymbol{\Gamma}}}_D - {}^{t}\underline{\underline{\mathbf{B}}} \right) : \left({}^{t}\underline{\underline{\boldsymbol{\Gamma}}}_D - {}^{t}\underline{\underline{\mathbf{B}}} \right)}} \tag{5.106a} $$

where from

$$ {}^{t}\dot{\bar{\varepsilon}}^{P} = \sqrt{\frac{2}{3}\, {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^{P} : {}^{t}\underline{\underline{\bar{\mathbf{d}}}}^{P}} \; . \tag{5.106b} $$

we obtain ${}^{t}\dot{\lambda} = {}^{t}\dot{\bar{\varepsilon}}^{P}$.

▶ The incremental formulation

If we assume a conservative loading and a fixed intermediate configuration, in order to satisfy the equilibrium equations at time (load level) $t + \triangle t$ we have to fulfill the *Principle of Minimum Potential Energy* (see Chap. 6):

$$ \delta\; {}^{t+\triangle t}_{\circ}\Pi \left({}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}^{E} \right) = 0 \tag{5.107} $$

where ${}^{t+\triangle t}_{\circ}\Pi$ is the potential energy corresponding to the $t+\triangle t$ configuration.

From Chap. 6, Eq. (6.51), we can write

$$ {}^{t+\triangle t}_{\circ}\Pi = \int_{{}^{\circ}V} {}^{\circ}\rho \; {}^{t+\triangle t}\mathtt{U} \; {}^{\circ}\mathrm{d}V \;+\; {}^{t+\triangle t}g \tag{5.108a} $$

where ${}^{\circ}V$: volume of the reference configuration; ${}^{t+\triangle t}\mathtt{U}$: elastic energy per unit mass stored in the $(t+\triangle t)$-configuration; and, ${}^{t+\triangle t}g$: potential of the external loads acting on the $(t+\triangle t)$-configuration.

Using the elastic free energy defined in Eq. (5.96b), we can write

$$ {}^{t+\triangle t}_{\circ}\Pi = \int_{{}^{\circ}V} {}^{t+\triangle t}\psi_e \; {}^{\circ}\mathrm{d}V \;+\; {}^{t+\triangle t}g \; . \tag{5.108b} $$

Equation (5.107) together with the above leads to

$$ \begin{aligned} \int_{{}^{\circ}V} {}^{t+\triangle t}\underline{\underline{\mathbf{\Gamma}}} : \delta\left({}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}^{E}\right) {}^{\circ}\mathrm{d}V \;=\; & \int_{{}^{t+\triangle t}V} {}^{t+\triangle t}\underline{\mathbf{f}}_b \cdot \delta\underline{\mathbf{u}} \; {}^{t+\triangle t}\mathrm{d}V \\ & + \int_{{}^{t+\triangle t}S} {}^{t+\triangle t}\underline{\mathbf{f}}_s \cdot \delta\underline{\mathbf{u}} \; {}^{t+\triangle t}\mathrm{d}S \; . \end{aligned} \tag{5.108c} $$

In the above equation:
${}^{t+\triangle t}V$: volume of the spatial configuration,
${}^{t+\triangle t}S$: external surface of the spatial configuration,
${}^{t+\triangle t}\underline{\mathbf{f}}_b$: body forces per unit volume, acting on the $(t+\triangle t)$-configuration,
${}^{t+\triangle t}\underline{\mathbf{f}}_s$: surface forces acting on the $(t+\triangle t)$-configuration,
$\underline{\mathbf{u}}$: displacement from t to $(t+\triangle t)$.

Also,

$$ \delta\,{}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}^{E} = \frac{\partial\,{}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}^{E}}{\partial\,{}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}} : \delta\,{}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}} \; . \tag{5.109a} $$

For the calculation of the fourth-order tensor $\dfrac{\partial\,{}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}^{E}}{\partial\,{}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}}$, we follow the appendix in (Dvorkin, Pantuso & Repetto 1994) and we get

$$ \frac{\partial\,{}^{t}_{\circ}H^{E}{}_{IJ}}{\partial\,{}^{t}_{\circ}H_{KL}} = \frac{\partial\,{}^{t}_{\circ}H^{E}{}_{IJ}}{\partial\,{}^{t}_{\circ}C^{E}{}_{MN}} \frac{\partial\,{}^{t}_{\circ}C^{E}{}_{MN}}{\partial\,{}^{t}_{\circ}C_{RS}} \frac{\partial\,{}^{t}_{\circ}C_{RS}}{\partial\,{}^{t}_{\circ}H_{KL}} \; . \tag{5.110a} $$

To calculate the derivative in the first factor on the r.h.s. of the above equation, we use

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{H}}}^{E} = \ln\sqrt{{}^{t}_{\circ}\underline{\underline{\mathbf{C}}}^{E}} \tag{5.110b} $$

being $\underline{\mathbf{\Phi}}_I^T = [\alpha_I, \beta_I, \gamma_I]$ $(I = 1, 2, 3)$ the 3 eigenvectors of ${}^t_\circ\underline{\underline{\mathbf{C}}}^E$ and λ_I its 3 eigenvalues

$$\frac{\partial {}^t_\circ H^E_{IJ}}{\partial {}^t_\circ C^E_{MN}} = \frac{\partial {}^t_\circ H^E_{IJ}}{\partial \alpha_K} \frac{\partial \alpha_K}{\partial {}^t_\circ C^E_{MN}} + \frac{\partial {}^t_\circ H^E_{IJ}}{\partial \beta_K} \frac{\partial \beta_K}{\partial {}^t_\circ C^E_{MN}} + \frac{\partial {}^t_\circ H^E_{IJ}}{\partial \gamma_K} \frac{\partial \gamma_K}{\partial {}^t_\circ C^E_{MN}} + \frac{\partial {}^t_\circ H^E_{IJ}}{\partial \lambda_K} \frac{\partial \lambda_K}{\partial {}^t_\circ C^E_{MN}} \ . \tag{5.110c}$$

To calculate the derivative in the second factor on the r.h.s. of Eq. (5.110a), we use

$${}^t_\circ\underline{\underline{\mathbf{C}}}^E = \left({}^t_\circ\underline{\underline{\mathbf{X}}}^P\right)^{-T} \cdot {}^t_\circ\underline{\underline{\mathbf{C}}} \cdot \left({}^t_\circ\underline{\underline{\mathbf{X}}}^P\right)^{-1} \ , \tag{5.110d}$$

and to calculate the derivative in the third factor we use

$${}^t_\circ\underline{\underline{\mathbf{H}}} = \ln \sqrt{{}^t_\circ\underline{\underline{\mathbf{C}}}} \ , \tag{5.110e}$$

and proceed as in Eq. (5.110c) but using the eigenvectors and eigenvalues of ${}^t_\circ\underline{\underline{\mathbf{C}}}$.

For the elastic deformation, we use

$${}^{t+\triangle t}\underline{\underline{\mathbf{\Gamma}}} = \underline{\underline{\underline{\underline{\mathbf{C}}}}}^E : {}^{t+\triangle t}_{\circ}\underline{\underline{\mathbf{H}}}^E \ , \tag{5.111}$$

where $\underline{\underline{\underline{\underline{\mathbf{C}}}}}^E$ is a constant and isotropic elasticity tensor (Hooke's law).

The main aspects of the finite strain elastoplastic formulation that we presented in this Section are:

- We use Lee's multiplicative decomposition of the deformation gradient.
- We use a hyperelastic constitutive relation for the *elastic strains/stress* relation; that is to say, we relate total strains with total stresses.
- We describe the plastic flow using the maximum dissipation principle (associated plasticity).
- We use the total Hencky strain as our strain measure. The reason for this choice is that according to the experimental data reported in (Anand 1979), *"the classical strain energy function of infinitesimal isotropic elasticity is in good agreement with experiment for a wide class of materials for moderately large deformations, provided the infinitesimal strain measure occurring in the strain energy function is replaced by the Hencky or logarithmic measure of finite strain"*.

From a numerical viewpoint, an additional advantage in using the Hencky strain measure is that the first invariant of the logarithmic strain tensor is the logarithmic volume strain; therefore many techniques developed for handling incompressibility in the infinitesimal strain problem can be carried over for the finite strain problem.

We discussed the above formulation, the "*total Lagrangian - Hencky formulation*" and its finite element implementation in (Dvorkin 1995a\1995b\1995c, Dvorkin & Assanelli 2000, Dvorkin, Pantuso & Repetto 1992\1993\1994\1995).

5.3 Constitutive relations in solid mechanics: thermoelastoplastic formulations

5.3.1 The isotropic thermoelastic constitutive model

Considering a hyperelastic solid under mechanical loads and thermal evolution, the two point variables that define the state of the solid at any instant are a strain measure and the temperature. Therefore, the stresses at any point in the solid are a function of the strains and temperature at that point (local action).

Using, for example, the Green-Lagrange strain tensor; and ${}^tT$ being the temperature, we can write for any particle in the spatial configuration its internal energy per unit mass (elastic energy + caloric energy) as

$$ {}^t\mathtt{U} = {}^t\mathtt{U}\left({}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}}, {}^tT\right) \tag{5.112a} $$

and considering a *reversible process* (Boley & Weiner 1960), we can write

$$ {}^t\eta = {}^t\eta\left({}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}}, {}^tT\right) \tag{5.112b} $$

where ${}^t\eta$ is the spatial *entropy per unit mass.*

The principle of energy conservation (First Law of Thermodynamics) can be written as

$$ \frac{D\,{}^t\mathtt{U}}{Dt} = \frac{1}{{}^{\circ}\rho}\,{}^t_{\circ}\underline{\underline{\mathbf{S}}} : {}^t_{\circ}\dot{\underline{\underline{\boldsymbol{\varepsilon}}}} + {}^tT\,{}^t\dot{\eta}\ . \tag{5.113} $$

We can define Helmholtz's free energy per unit mass, (Boley & Weiner 1960, Malvern 1969) as:

$$ {}^t\psi\left({}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}}, {}^tT\right) = {}^t\mathtt{U}\left({}^t_{\circ}\underline{\underline{\boldsymbol{\varepsilon}}}, {}^tT\right) - {}^tT\,{}^t\eta\ . \tag{5.114a} $$

Using the above and Eq. (5.113), we get

$$ \left(\frac{\partial^t\psi}{\partial^t_{\circ}\varepsilon_{IJ}} - \frac{1}{{}^{\circ}\rho}\,{}^t_{\circ}S^{IJ}\right){}^t_{\circ}\dot{\varepsilon}_{IJ} + \left(\frac{\partial^t\psi}{\partial^tT} + {}^t\eta\right){}^t\dot{T} = 0\ . \tag{5.114b} $$

Hence considering the isothermal (${}^t\dot{T} = 0$) and isometric (${}^t_{\circ}\dot{\varepsilon}_{IJ} = 0$) cases, we have

$$ {}^t_{\circ}S^{IJ} = {}^{\circ}\rho\frac{\partial^t\psi}{\partial^t_{\circ}\varepsilon_{IJ}} \tag{5.114c} $$

$$ {}^t\eta = -\frac{\partial^t\psi}{\partial^tT}\ . \tag{5.114d} $$

In a hyperelastic material the stresses and entropy are a function of the strain/temperature value at the point and not of its history; then, the above equations are valid for any process.

Since the free energy, ${}^{t}\psi$, is invariant under changes of reference frame, for an *isotropic material,* it can only depend on the invariants of ${}^{t}_{\circ}\underline{\underline{\varepsilon}}$, which for the t-configuration, and considering the strain tensor Cartesian components, are

$$ {}^{t}_{\circ}I_1 = {}^{t}_{\circ}\varepsilon_{\alpha\alpha} \tag{5.115a} $$

$$ {}^{t}_{\circ}I_2 = \frac{1}{2}\,{}^{t}_{\circ}\varepsilon_{\alpha\beta}\,{}^{t}_{\circ}\varepsilon_{\alpha\beta} - \frac{1}{2}\left({}^{t}_{\circ}I_1\right)^2 \tag{5.115b} $$

$$ {}^{t}_{\circ}I_3 = \frac{1}{6}\,e_{\alpha\beta\gamma}\,e_{\delta\epsilon\zeta}\,{}^{t}_{\circ}\varepsilon_{\alpha\delta}\,{}^{t}_{\circ}\varepsilon_{\beta\epsilon}\,{}^{t}_{\circ}\varepsilon_{\gamma\zeta}\ . \tag{5.115c} $$

Hence,

$$ {}^{t}\psi = {}^{t}\psi\left({}^{t}_{\circ}I_1,\ {}^{t}_{\circ}I_2,\ {}^{t}_{\circ}I_3,\ {}^{t}T\right)\ . \tag{5.116} $$

The most general expression for the free energy is

$$ \begin{aligned} {}^{t}\psi &= a_0 + a_1\,{}^{t}_{\circ}I_1 + a_2\,{}^{t}_{\circ}I_2 + a_3\,{}^{t}_{\circ}I_3 + a_4\,\left({}^{t}T - T_R\right) + a_5\left({}^{t}T - T_R\right)^2 \\ &\quad + a_6\,{}^{t}_{\circ}I_1\,\left({}^{t}T - T_R\right) + a_7\,{}^{t}_{\circ}I_2\,\left({}^{t}T - T_R\right) + a_8\,{}^{t}_{\circ}I_3\,\left({}^{t}T - T_R\right) \\ &\quad + a_9\left({}^{t}_{\circ}I_1\right)^2 + \dots\ . \end{aligned} \tag{5.117} $$

In the above equation T_R is the *reference temperature*, usually the temperature of the undeformed solid.

Using Eq. (5.114c), we get

$$ {}^{t}_{\circ}S_{\alpha\beta} = {}^{\circ}\rho\frac{\partial^{t}\psi}{\partial^{t}_{\circ}I_i}\frac{\partial^{t}_{\circ}I_i}{\partial^{t}_{\circ}\varepsilon_{\alpha\beta}}\ , \tag{5.118a} $$

where

$$ \frac{\partial^{t}_{\circ}I_1}{\partial^{t}_{\circ}\varepsilon_{\alpha\beta}} = \delta_{\alpha\beta}\ , \tag{5.118b} $$

$$ \frac{\partial^{t}_{\circ}I_2}{\partial^{t}_{\circ}\varepsilon_{\alpha\beta}} = {}^{t}_{\circ}\varepsilon_{\alpha\beta} - {}^{t}_{\circ}I_1\,\delta_{\alpha\beta}\ , \tag{5.118c} $$

$$ \frac{\partial^{t}_{\circ}I_3}{\partial^{t}_{\circ}\varepsilon_{\alpha\beta}} = \frac{1}{6}e_{\alpha\gamma\delta}\,e_{\beta\epsilon\zeta}\,{}^{t}_{\circ}\varepsilon_{\gamma\epsilon}\,{}^{t}_{\circ}\varepsilon_{\delta\zeta}\ . \tag{5.118d} $$

Physical restrictions used to determine the material constants:

- $\begin{matrix} {}^{t}_{\circ}\varepsilon_{\alpha\beta} = 0 \\ {}^{t}T = T_R \end{matrix} \quad \Longrightarrow \quad {}^{t}_{\circ}S_{\alpha\beta} = 0$; hence, $a_1 = 0.$
- For the above condition, we set an arbitrary value for the free energy; hence, we set $a_0 = 0$.
- Doing the same consideration for the entropy, we set $a_4 = 0$.

► **Specialization for small strains and small temperature increments**

In this case:

$$
\begin{aligned}
{}^{t}_{\circ}\varepsilon_{\alpha\beta} &\longrightarrow {}^{t}\varepsilon_{\alpha\beta} \qquad (\textit{infinitesimal strain tensor}) \\
{}^{t}_{\circ}S_{\alpha\beta} &\longrightarrow {}^{t}\sigma_{\alpha\beta} \qquad (\textit{Cauchy stress tensor}) \ .
\end{aligned}
$$

Using Eq. (5.114c) and (5.117) and neglecting higher-order terms in ${}^{t}\varepsilon_{\alpha\beta}$ and $({}^{t}T - T_R)$, we get, for an isotropic material (Malvern 1969),

$$
{}^{t}\sigma_{\alpha\beta} = \lambda\ \delta_{\alpha\beta}\ {}^{t}\varepsilon_{\gamma\gamma} + 2\ G\ {}^{t}\varepsilon_{\alpha\beta} - \frac{E\ \alpha}{(1-2\nu)}\ \left({}^{t}T - T_R\right)\ \delta_{\alpha\beta} \qquad (5.119)
$$

where E : Young's modulus; ν : Poisson's ratio; G : shear modulus, $G = \frac{E}{2(1+\nu)}$; $\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$; and, α : linear coefficient of thermal expansion.

For an isotropic solid with constant and uniform material properties, in the absence of volumetric heat generation or consumption, the heat conduction equation is (Boley & Weiner 1960):

$$
k\ \nabla^2\ {}^{t}T = {}^{\circ}\rho\ {}^{t}T\ {}^{t}\dot{\eta} \qquad (5.120)
$$

where k is the heat conduction coefficient of the material (Fourier's law).

The above equation together with Eq. (5.114d) leads to,

$$
k\ \nabla^2\ {}^{t}T = -\ {}^{\circ}\rho\ {}^{t}T\ \left[\frac{\partial^2\ {}^{t}\psi}{\partial^{t}T\ \partial^{t}\varepsilon_{\alpha\beta}}\ {}^{t}\dot{\varepsilon}_{\alpha\beta} + \frac{\partial^2\ {}^{t}\psi}{\partial^{t}T^2}\ {}^{t}\dot{T}\right] \ . \qquad (5.121)
$$

The specific heat of the material is defined as:

$$
{}^{t}c = \frac{\partial^{t}\mathtt{U}}{\partial^{t}T} \qquad (5.122)
$$

and after some algebra,

$$
{}^{t}c = {}^{t}T\ \frac{\partial^{t}\eta}{\partial^{t}T} \ . \qquad (5.123)
$$

Using Eqs. (5.114c) and (5.114d) in (5.121), we get

$$
k\ \nabla^{2t}T = -\ {}^{\circ}\rho\ {}^{t}T\ \frac{\partial}{\partial^{t}T}\left(\frac{{}^{t}\sigma_{\alpha\beta}}{{}^{\circ}\rho}\right)\ {}^{t}\dot{\varepsilon}_{\alpha\beta} + {}^{\circ}\rho\ {}^{t}T\ \frac{\partial^{t}\eta}{\partial^{t}T}\ {}^{t}\dot{T} \ . \qquad (5.124)
$$

And using Eqs. (5.119) and (5.123) in (5.124), we get

$$
k\ \nabla^{2t}T = \frac{{}^{t}T\ E\ \alpha}{(1-2\nu)}\ {}^{t}\dot{\varepsilon}_{\alpha\alpha} + {}^{\circ}\rho\ {}^{t}c\ {}^{t}\dot{T} \ . \qquad (5.125)
$$

Equations (5.119) and (5.125) show that in the most general case of thermoelastic materials the heat transfer and stress analysis problems are *coupled*.

However, in most engineering applications the first term on the r.h.s. of Eq. (5.125) can be neglected.

In (Boley & Weiner 1960) numerical examples in aluminum and steel were considered and it was shown that the coupling is negligible when

$$\frac{{}^t\dot{\varepsilon}_{\alpha\alpha}}{3\ \alpha\ {}^t\dot{T}} \ll 20\ . \tag{5.126}$$

5.3.2 A thermoelastoplastic constitutive model

In Sect. 5.2.5 we developed the constitutive relation for an elastoplastic material under infinitesimal strains; in that section we made the assumption of a purely mechanical formulation, now we shall remove that limitation incorporating thermal effects. However, we will keep the other assumptions made in that section: infinitesimal strains, limit loading condition (yield criterion), rate-independent behavior, stable material, etc.

Since we have to take into account the thermal strains, we rewrite Eq. (5.38) as:

$$ {}^t\underline{\underline{\mathbf{d}}} = {}^t\underline{\underline{\mathbf{d}}}^E + {}^t\underline{\underline{\mathbf{d}}}^P + {}^t\underline{\underline{\mathbf{d}}}^{TH}\ . \tag{5.127}$$

For the yield condition, we rewrite Eq. (5.48) as:

$$ {}^tf\left({}^t\underline{\underline{\boldsymbol{\sigma}}}\ ,\ {}^tq_i\ \ i=1,n\ ,\ {}^tT\right) = 0\ . \tag{5.128}$$

Considering that we are focusing on the behavior of metals, we will keep on using the von Mises yield criterion and therefore in the case of isotropic hardening, we can rewrite Eq. (5.55) as:

$$ {}^tf = \frac{1}{2}\ {}^t\underline{\underline{\mathbf{s}}}\ :\ {}^t\underline{\underline{\mathbf{s}}} - \frac{{}^t\sigma_y^2}{3} = 0 \tag{5.129a}$$

where ${}^t\sigma_y = {}^t\sigma_y\left({}^t\bar{\varepsilon}^P,\ {}^tT\right)$.

In the case of kinematic hardening we can rewrite Eq. (5.55) as (Snyder 1980):

$$ {}^tf = \frac{1}{2}\ \left({}^t\underline{\underline{\mathbf{s}}} - {}^t\underline{\underline{\boldsymbol{\alpha}}}\right)\ :\ \left({}^t\underline{\underline{\mathbf{s}}} - {}^t\underline{\underline{\boldsymbol{\alpha}}}\right) - \frac{{}^t\sigma_y^2}{3} \tag{5.129b}$$

where ${}^t\sigma_y = {}^t\sigma_y\left({}^tT\right)$; ${}^t\alpha_{ij} = \int_\circ^t {}^\tau\dot{\alpha}_{ij}\ d\tau$; and, ${}^t\dot{\alpha}_{ij} = {}^tc\left({}^tT\right)\ {}^td_{ij}^P$.

For the unstressed state with no previously accumulated plastic strains (Boley & Weiner 1960)

$$ {}^tf(\underline{\underline{\mathbf{0}}}, 0, {}^tT) < 0\ . \tag{5.130}$$

During plastic loading the *consistency equation* takes the form (Boley & Weiner 1960),

$$ \dot{f} = \frac{\partial^t f}{\partial^t s^{ij}}\ \dot{s}^{ij} + \frac{\partial^t f}{\partial^t \varepsilon_{ij}^P}\ {}^td_{ij}^P + \frac{\partial^t f}{\partial^t T}\ {}^t\dot{T} = 0\ . \tag{5.131}$$

Following (Boley & Weiner 1960) we will study three possible cases:

(a)

$$\frac{\partial^t f}{\partial^t s^{ij}}\ {}^t\dot{s}^{ij} \ + \ \frac{\partial^t f}{\partial^t T}\ {}^t\dot{T} = 0 \ . \tag{5.132a}$$

Using the above equation together with Eq. (5.131), we see that it is possible for the stress components and the temperature to change so that the point remains on the yield surface but without further plasticity development. Therefore this change is called *neutral* $\left({}^t d^P_{ij} = 0\right)$.

(b)

$$\frac{\partial^t f}{\partial^t s^{ij}}\ {}^t\dot{s}^{ij} \ + \ \frac{\partial^t f}{\partial^t T}\ {}^t\dot{T} < 0 \ . \tag{5.132b}$$

The above equation represents an *unloading* and therefore ${}^t d^P_{ij} = 0$.

(c)

$$\frac{\partial^t f}{\partial^t s^{ij}}\ {}^t\dot{s}^{ij} \ + \ \frac{\partial^t f}{\partial^t T}\ {}^t\dot{T} > 0 \ . \tag{5.132c}$$

The above equation represents a condition of plastic loading and using Eq.(5.131), we get ${}^t d^P_{ij}$.

► Stress - strain - temperature relation

As we did above for the case of isothermal plasticity we consider that the *plastic flow* maximizes the *plastic dissipation* (Lubliner 1985).

We again define the plastic dissipation via Eq. (5.57) and imposing during *plastic loading* the condition ${}^t f = 0$ we obtain, in a general curvilinear system,

$$\mathrm{d}\sigma^{ij}\ {}^t d^P_{ij} \ = \ {}^t\dot{\lambda}\ \left[\frac{\partial^t f}{\partial^t \sigma^{ij}}\ \mathrm{d}\sigma^{ij} \ + \ \frac{\partial^t f}{\partial^t T}\ \mathrm{d}T\right] \ . \tag{5.133}$$

Considering Eq. (5.132c) and since ${}^t\dot{\lambda} \geqslant 0$, we get for plastic loading the condition

$${}^t\dot{\sigma}^{ij}\ {}^t d^P_{ij} > 0 \ , \tag{5.134}$$

which is the same stability condition already obtained for the isothermal case (Drucker's postulate). Again, we obtain the convexity condition for the yield surface in the stress space.

From Eq. (5.133),

$$\left[{}^t d^P_{ij} \ - \ {}^t\dot{\lambda}\ \frac{\partial^t f}{\partial^t \sigma^{ij}}\right]\ \mathrm{d}\sigma^{ij} \ + \ \frac{\partial^t f}{\partial^t T}\ \mathrm{d}T = 0 \ . \tag{5.135}$$

Hence, the normality rule already developed for the isothermal case applies:

$$ {}^t d_{ij}^P = {}^t\dot{\lambda} \frac{\partial^t f}{\partial^t \sigma^{ij}} . \tag{5.136} $$

Stress - strain relations for the case of isotropic hardening

Since we are considering the case of infinitesimal strains, we can write in a Cartesian system,

$$ \mathrm{d}\varepsilon_{\alpha\beta} = \mathrm{d}\varepsilon^E_{\alpha\beta} + \mathrm{d}\varepsilon^P_{\alpha\beta} + \mathrm{d}\varepsilon^{TH}_{\alpha\beta} \tag{5.137a} $$

and

$$ {}^t\sigma_{\alpha\beta} = {}^tC^E_{\alpha\beta\gamma\delta}\, {}^t\varepsilon^E_{\gamma\delta} . \tag{5.137b} $$

We consider an isotropic linear elastic material with elastic constants function of the temperature; therefore (Snyder 1980),

$$ \mathrm{d}\sigma_{\alpha\beta} = {}^tC^E_{\alpha\beta\gamma\delta}\, \mathrm{d}\varepsilon^E_{\gamma\delta} + \frac{\partial^t C^E_{\alpha\beta\gamma\delta}}{\partial^t T}\, {}^t\varepsilon^E_{\gamma\delta}\, \mathrm{d}T . \tag{5.138} $$

Hence,

$$ \mathrm{d}\sigma_{\alpha\beta} = {}^tC^E_{\alpha\beta\gamma\delta} \left[\mathrm{d}\varepsilon_{\gamma\delta} - \mathrm{d}\varepsilon^P_{\gamma\delta} - \mathrm{d}\varepsilon^{TH}_{\gamma\delta}\right] + \frac{\partial^t C^E_{\alpha\beta\gamma\delta}}{\partial^t T}\, {}^t\varepsilon^E_{\gamma\delta}\, \mathrm{d}T . \tag{5.139a} $$

For a von Mises material,

$$ \mathrm{d}\varepsilon^P_{\gamma\delta} = \mathrm{d}\lambda\, {}^ts_{\gamma\delta} \tag{5.139b} $$

and for an isotropic thermal expansion,

$$ \mathrm{d}\varepsilon^{TH}_{\gamma\delta} = {}^t\alpha\, \mathrm{d}T\, \delta_{\gamma\delta} . \tag{5.139c} $$

During plastic loading $df = 0$ and using Eq. (5.131),

$$ \frac{\partial^t f}{\partial^t \sigma_{\alpha\beta}} \mathrm{d}\sigma_{\alpha\beta} + \frac{\partial^t f}{\partial^t \bar{\varepsilon}^P} \mathrm{d}\bar{\varepsilon}^P + \frac{\partial^t f}{\partial^t T} \mathrm{d}T = 0 . \tag{5.140} $$

Developing each of the terms in the above equation, we obtain

$$ \begin{aligned} \frac{\partial^t f}{\partial^t \sigma_{\alpha\beta}} \mathrm{d}\sigma_{\alpha\beta} &= {}^ts_{\alpha\beta} \left[\, {}^tC^E_{\alpha\beta\gamma\delta} \left(\mathrm{d}\varepsilon_{\gamma\delta} - \mathrm{d}\lambda\, {}^ts_{\gamma\delta} - {}^t\alpha\, \mathrm{d}T\, \delta_{\gamma\delta}\right)\right. \\ &\quad \left. + \frac{\partial^t C^E_{\alpha\beta\gamma\delta}}{\partial^t T}\, {}^t\varepsilon^E_{\gamma\delta}\, \mathrm{d}T\right] && (5.141a) \\ \frac{\partial^t f}{\partial^t \bar{\varepsilon}^P} \mathrm{d}\bar{\varepsilon}^P &= -\frac{4}{9}\, {}^t\sigma_y^2 \frac{\partial^t \sigma_y}{\partial^t \bar{\varepsilon}^P} \mathrm{d}\lambda && (5.141b) \\ \frac{\partial^t f}{\partial^t T} \mathrm{d}T &= -\frac{2}{3}\, {}^t\sigma_y \frac{\partial^t \sigma_y}{\partial^t T} \mathrm{d}T , && (5.141c) \end{aligned} $$

therefore (Snyder 1980),

$$\mathrm{d}\lambda = \frac{{}^t s_{\alpha\beta} \, [\, {}^t C^E_{\alpha\beta\gamma\delta} \ (\mathrm{d}\varepsilon_{\gamma\delta} \ - \ {}^t\alpha \ \mathrm{d}T \ \delta_{\gamma\delta}) + \ \frac{\partial^t C^E_{\alpha\beta\gamma\delta}}{\partial^t T} \ {}^t\varepsilon^E_{\gamma\delta} \ \mathrm{d}T] \ - \ \frac{2}{3} \ {}^t\sigma_y \ \frac{\partial^t \sigma_y}{\partial^t T} \ \mathrm{d}T}{{}^t s_{\eta\theta} \ {}^t C^E_{\eta\theta\mu\xi} \ {}^t s_{\mu\xi} \ + \ \frac{4}{9} \ {}^t\sigma_y^2 \ \frac{\partial^t \sigma_y}{\partial^t \bar{\varepsilon}^P}} . \tag{5.142}$$

Hence, we introduce the above in Eqs. (5.139a-5.139c) and we can inmediately relate increments in strains/temperature with stress increments.

In order to be able to evaluate the terms in Eq. (5.142) it is necessary to relate $\left(\frac{\partial^t \sigma_y}{\partial^t \bar{\varepsilon}^P}\right)$ and $\left(\frac{\partial^t \sigma_y}{\partial^t T}\right)$ to the actual material behavior (Snyder 1980).

From the data obtained in isothermal tensile tests of virgin samples, we can develop the idealized bilinear stress-strain curves shown in Fig. 5.11.

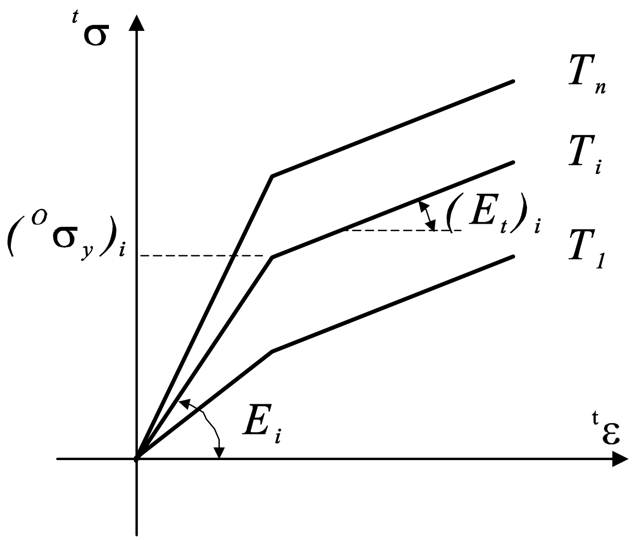


Fig. 5.11. Stress-strain curves at different temperatures, T_i

For a constant temperature curve, we can write

$$ {}^t\sigma_y = ({}^\circ\sigma_y)_T \ + \ \left[{}^t\varepsilon \ - \ \left(\frac{{}^\circ\sigma_y}{E}\right)_T \right] \ (E_t)_T \tag{5.143a}$$

$$ {}^t\varepsilon = \ {}^t\varepsilon^P \ + \ \frac{{}^t\sigma_y}{(E)_T} \ . \tag{5.143b}$$

Therefore,

$$ {}^t\sigma_y = ({}^\circ\sigma_y)_T \ + \ {}^t\varepsilon^P \ \frac{(E \ E_t)_T}{(E \ - \ E_t)_T} \ . \tag{5.143c}$$

Using, as in the isothermal case, the concept of a *universal stress-strain curve* that is valid for any multiaxial stress-strain state, we can use Eq. (5.143c) for any stress - strain state, provided that ${}^t\varepsilon^P$ is replaced by ${}^t\overline{\varepsilon}^P$ (Eq. (5.74a)). Hence, in Eq. (5.142), we have

$$\frac{\partial^t\sigma_y}{\partial^t\overline{\varepsilon}^P} = \left(\frac{E\ E_t}{E\ -\ E_t}\right)_T \tag{5.144a}$$

$$\frac{\partial^t\sigma_y}{\partial^t T} = \left(\frac{\partial^\circ\sigma_y}{\partial^t T}\right)_T + {}^t\overline{\varepsilon}^P \left[\frac{\partial}{\partial^t T}\left(\frac{E\ E_t}{E\ -\ E_t}\right)\right]_T . \tag{5.144b}$$

Hence, we can rewrite Eq. (5.142) as:

$$\mathrm{d}\lambda = \frac{{}^t s_{\alpha\beta}\left[\ {}^tC^E_{\alpha\beta\gamma\delta}\ \left(\mathrm{d}\varepsilon_{\gamma\delta}\ -\ {}^t\alpha\ \mathrm{d}T\ \delta_{\gamma\delta}\right) + \frac{\partial^t C^E_{\alpha\beta\gamma\delta}}{\partial^t T}\ {}^t\varepsilon^E_{\gamma\delta}\ \mathrm{d}T\ \right]}{{}^t s_{\eta\theta}\ {}^tC^E_{\eta\theta\mu\xi}\ {}^t s_{\mu\xi}\ +\ \frac{4}{9}\ {}^t\sigma_y^2\ \left(\frac{E\ E_t}{E\ -\ E_t}\right)_T}$$

$$-\ \frac{\frac{2}{3}\ {}^t\sigma_y\ \left[\left(\frac{\partial^\circ\sigma_y}{\partial^t T}\right)_T\ +\ {}^t\overline{\varepsilon}^P\left(\frac{\partial}{\partial^t T}\left(\frac{E\ E_t}{E\ -\ E_t}\right)\right)_T\right]\ \mathrm{d}T}{{}^t s_{\eta\theta}\ {}^tC^E_{\eta\theta\mu\xi}\ {}^t s_{\mu\xi}\ +\ \frac{4}{9}\ {}^t\sigma_y^2\ \left(\frac{E\ E_t}{E\ -\ E_t}\right)_T} . \tag{5.145}$$

In the above equation, we consider a linear isotropic elastic model; therefore using Eqs. (5.15) and (5.16), we get

$${}^tC^E_{\alpha\beta\gamma\delta} = (\lambda)_T\,\delta_{\alpha\beta}\delta_{\gamma\delta} + (G)_T\,(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \tag{5.146}$$

and taking into account that ${}^t s_{\alpha\alpha} = 0$, we get

$${}^t s_{\alpha\beta}\ {}^tC^E_{\alpha\beta\gamma\delta} = 2\ \ (G)_T\ \ {}^t s_{\gamma\delta}\ . \tag{5.147}$$

Taking into account that (Snyder 1980)

$$\left[{}^t\left(C^E\right)^{-1}\right]_{\alpha\beta\gamma\delta} = -\left(\frac{\nu}{E}\right)_T\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{4\,(G)_T}\,(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \tag{5.148}$$

we can show that

$${}^t s_{\alpha\beta}\ \ \frac{\partial^t C^E_{\alpha\beta\gamma\delta}}{\partial^t T}\ {}^t\varepsilon^E_{\gamma\delta} = \left[\frac{1}{G}\left(\frac{\partial G}{\partial^t T}\right)\right]_T\ {}^t s_{\alpha\beta}\ {}^t\sigma_{\alpha\beta}\ . \tag{5.149}$$

Hence,

$$\mathrm{d}\lambda = \frac{2\,(G)_T\ \ {}^t s_{\gamma\delta}\ \mathrm{d}\varepsilon_{\gamma\delta} + \left(\frac{1}{G}\frac{\partial G}{\partial T}\right)_T\ {}^t s_{\alpha\beta}\ {}^t s_{\alpha\beta}\ \mathrm{d}T}{\frac{4}{3}\,(G)_T\ \ \left({}^t\sigma_y\right)^2 + \frac{4}{9}\ {}^t\sigma_y^2\ \left(\frac{E\ E_t}{E-E_t}\right)_T}$$

$$-\ \frac{\frac{2}{3}\ {}^t\sigma_y\left[\left(\frac{\partial^0\sigma_y}{\partial T}\right)_T +\ {}^t\overline{\varepsilon}^P\left(\frac{\partial}{\partial^t T}\left(\frac{E\ E_t}{E-E_t}\right)\right)_T\right]\mathrm{d}T}{\frac{4}{3}\,(G)_T\ \ \left({}^t\sigma_y\right)^2 + \frac{4}{9}\ {}^t\sigma_y^2\ \left(\frac{E\ E_t}{E-E_t}\right)_T} . \tag{5.150}$$

Stress - strain relations for the case of kinematic hardening

We use the kinematic hardening results in Section 5.2.5 and adapt them for the case of nonisothermal processes,

$$ {}^{t}f = \frac{1}{2}\left({}^{t}s_{\gamma\delta} - {}^{t}\alpha_{\gamma\delta}\right)\left({}^{t}s_{\gamma\delta} - {}^{t}\alpha_{\gamma\delta}\right) - \frac{1}{3}\,{}^{t}\sigma_y^2 = 0 \quad (5.151a)$$

$$ {}^{t}\sigma_y = {}^{t}\sigma_y\left({}^{t}T\right) \quad (5.151b)$$

$$ {}^{t}\alpha_{\gamma\delta} = \int_0^t {}^{t}\dot{\alpha}_{\gamma\delta}\,\mathrm{d}t \quad (5.151c)$$

$$ {}^{t}\dot{\alpha}_{\gamma\delta} = {}^{t}c\,{}^{t}d^{P}_{\gamma\delta} \quad (5.151d)$$

$$ {}^{t}c = {}^{t}c\left({}^{t}T\right) \; . \quad (5.151e)$$

During the plastic loading

$$ \frac{\partial^t f}{\partial^t \sigma_{\gamma\delta}} d\sigma_{\gamma\delta} + \frac{\partial^t f}{\partial^t \alpha_{\gamma\delta}} d\alpha_{\gamma\delta} + \frac{\partial^t f}{\partial^t \sigma_y} d\sigma_y = 0 \; . \quad (5.152)$$

Developing each of the terms in the above equation, we obtain

$$ \frac{\partial^t f}{\partial^t \sigma_{\gamma\delta}} \mathrm{d}\sigma_{\gamma\delta} = \left({}^{t}s_{\gamma\delta} - {}^{t}\alpha_{\gamma\delta}\right) \quad (5.153a)$$

$$ \left\{ {}^{t}C^{E}_{\gamma\delta\varphi\xi}\left[\mathrm{d}\varepsilon_{\varphi\xi} - \mathrm{d}\lambda\left({}^{t}s_{\varphi\xi} - {}^{t}\alpha_{\varphi\xi}\right) - {}^{t}\alpha\,\mathrm{d}T\,\delta_{\varphi\xi}\right] + \frac{\partial^t C^{E}_{\gamma\delta\varphi\xi}}{\partial T}\,{}^{t}\varepsilon^{E}_{\varphi\xi}\,\mathrm{d}T \right\}$$

$$ \frac{\partial^t f}{\partial^t \alpha_{\gamma\delta}} \mathrm{d}\alpha_{\gamma\delta} = -\left({}^{t}s_{\gamma\delta} - {}^{t}\alpha_{\gamma\delta}\right){}^{t}c\,\mathrm{d}\lambda\left({}^{t}s_{\gamma\delta} - {}^{t}\alpha_{\gamma\delta}\right) = -\frac{2}{3}\,{}^{t}c\,\mathrm{d}\lambda\,{}^{t}\sigma_y^2 \quad (5.153b)$$

$$ \frac{\partial^t f}{\partial^t \sigma_y} \mathrm{d}\sigma_y = -\frac{2}{3}\,{}^{t}\sigma_y\,\frac{\partial^t \sigma_y}{\partial^t T}\,\mathrm{d}T \quad (5.153c)$$

therefore (Snyder 1980),

$$ \mathrm{d}\lambda = \frac{\left({}^{t}s_{\gamma\delta} - {}^{t}\alpha_{\gamma\delta}\right)\left[{}^{t}C^{E}_{\gamma\delta\varphi\xi}\left(\mathrm{d}\varepsilon_{\varphi\xi} - {}^{t}\alpha\,\mathrm{d}T\,\delta_{\varphi\xi}\right) + \frac{\partial^t C^{E}_{\gamma\delta\varphi\xi}}{\partial T}\,{}^{t}\varepsilon^{E}_{\varphi\xi}\,\mathrm{d}T\right]}{\left({}^{t}s_{\eta\theta} - {}^{t}\alpha_{\eta\theta}\right){}^{t}C^{E}_{\eta\theta\mu\psi}\left({}^{t}s_{\mu\psi} - {}^{t}\alpha_{\mu\psi}\right) + \frac{2}{3}\,{}^{t}c\,{}^{t}\sigma_y^2}$$

$$ - \frac{\frac{2}{3}\,{}^{t}\sigma_y\,\frac{\partial^t \sigma_y}{\partial^t T}\,\mathrm{d}T}{\left({}^{t}s_{\eta\theta} - {}^{t}\alpha_{\eta\theta}\right){}^{t}C^{E}_{\eta\theta\mu\psi}\left({}^{t}s_{\mu\psi} - {}^{t}\alpha_{\mu\psi}\right) + \frac{2}{3}\,{}^{t}c\,{}^{t}\sigma_y^2} \quad (5.154)$$

Again, as in the case of isotropic hardening, we relate the above expression to the actual material behavior using the information contained in the isothermal uniaxial stress-strain curves.

For an isothermal loading in a bi-linear material, we can use the result in Example 5.13 and obtain,

$$ {}^{t}c(T) = \frac{2}{3} \left(\frac{E\ E_t}{E - E_t} \right)_T \quad . \tag{5.155} $$

5.4 Viscoplasticity

In Sects. 5.2 and 5.3, we discussed constitutive relations that have a common feature: the response of the solids is *instantaneous*; that is to say, when a load is applied, either a mechanical or a thermal load, the solid instantaneously develops the corresponding displacements and strains.

We know, from our experience, that this is not the case in many situations; e.g. a metallic structure under elevated temperature increases its deformation with time; a concrete structure in the first few months after it has been cast increases its deformation with time, etc.

There is also an important experimental observation related to the response of materials, in particular metals, to rapid loads: the apparent yield stress increases with the deformation velocity. In the previous sections, when considering instantaneous plasticity, we represented the strain hardening of metals with equations of the form:

$$ \sigma_y = \sigma_y(\overline{\varepsilon}, T) \ . \tag{5.156} $$

To take into account the above commented experimental observation, the yield stress has to present the following functional dependence (Backofen 1972):

$$ \sigma_y = \sigma_y(\overline{\varepsilon}, \dot{\overline{\varepsilon}}, T) \ . \tag{5.157} $$

We can say that the strain-rate effect shown in Eq.(5.157) is a *viscous* effect. There are basically two ways in which a viscous effect can enter a solid's constitutive relation:

- In the *viscoelastic* constitutive relations, the elastic part of the solid deformation presents viscous effects. In this book, we are not going to discuss this kind of constitutive relations and we refer the readers to (Pipkin 1972) for a detailed discussion.
- In the *viscoplastic* constitutive relations (Perzyna 1966), the permanent deformation presents viscous effects. The examples we discussed above are described using viscoplastic constitutive relations and also, other important problems like metal-forming processes are very well described using this constitutive theory (Zienkiewicz, Jain & Oñate 1977, Kobayashi, Oh & Altan 1989).

As in the case of elastoplasticity, we can divide the total strain rate into its elastic and viscoplastic parts; hence, we get an equation equivalent to Eq. (5.38), but now for an *elastoviscoplastic* solid:

$$ {}^{t}\underline{\underline{\mathbf{d}}} = {}^{t}\underline{\underline{\mathbf{d}}}^{E} + {}^{t}\underline{\underline{\mathbf{d}}}^{VP} \tag{5.158}$$

where, ${}^{t}\underline{\underline{\mathbf{d}}}^{VP}$ is the viscoplastic strain rate tensor.

In some cases, for example when modeling bulk metal-forming processes (Zienkiewicz, Jain & Oñate 1977), ${}^{t}\underline{\underline{\mathbf{d}}}^{E} << {}^{t}\underline{\underline{\mathbf{d}}}^{VP}$. Therefore, we can set ${}^{t}\underline{\underline{\mathbf{d}}}^{E} = \underline{\underline{\mathbf{0}}}$, introducing a very important simplification in the model without any significant loss in accuracy; these are the *rigid-viscoplastic* material models.

▶ The yield surface

As in the case of plasticity, a yield surface is defined in the stress space with an equation identical to Eq. (5.48):

$$ {}^{t}f\ ({}^{t}\underline{\underline{\boldsymbol{\sigma}}},\ {}^{t}q_i \quad i = 1, n) = 0\ . \tag{5.159}$$

The internal variables ${}^{t}q_i$ indicate that in the viscoplastic case, the yield surface is also modified in its shape and/or position by the hardening phenomenon.

In the case of elastoplastic material models, we remember that Eqs. (5.49a-5.49b) established that in the stress space every point in the solid is either *inside the yield surface* (${}^{t}f < 0$ and therefore the behavior is elastic and ${}^{t}\underline{\underline{\mathbf{d}}}^{P} = \underline{\underline{\mathbf{0}}}$) or *on the yield surface* (${}^{t}f = 0$ and therefore the behavior is elastoplastic and permanent deformations are generated with ${}^{t}\underline{\underline{\mathbf{d}}}^{P} \neq \underline{\underline{\mathbf{0}}}$).

In the viscoplastic theory, the point can be either *inside the yield surface* (${}^{t}f < 0$ and therefore ${}^{t}\underline{\underline{\mathbf{d}}}^{VP} = \underline{\underline{\mathbf{0}}}$) or *outside the yield surface* (${}^{t}f > 0$ and in this case ${}^{t}\underline{\underline{\mathbf{d}}}^{VP} \neq \underline{\underline{\mathbf{0}}}$).

▶ The flow rule

In a Cartesian system, for viscoplastic materials, we use the following flow rule (Perzyna 1966):

$$ {}^{t}d^{VP}_{\alpha\beta} = \gamma \frac{\partial\, {}^{t}f}{\partial\, {}^{t}\sigma_{\alpha\beta}} \ \langle \phi\ ({}^{t}f) \rangle\ . \tag{5.160}$$

In the above equation, we use the Macauley brackets defined by:

$$ \langle a \rangle = a \quad if\ a > 0 \tag{5.161a}$$

$$ \langle a \rangle = 0 \quad if\ a \leq 0\ . \tag{5.161b}$$

An important difference between the flow rate for the viscoplastic constitutive model (Eq. (5.160)) and the flow rate for the plastic constitutive model (Eq. (5.60)) is that in the present case, γ the *fluidity parameter* is a material constant, while in the plasticity theory ${}^{t}\dot{\lambda}$ is a flow constant, derived by imposing the consistency condition during the plastic loading.

Obviously, the correct value of γ and the correct expression for $\phi\ ({}^{t}f)$ are derived from experimental observations.

In what follows we will concentrate on the details of a rigid-viscoplastic relation suited for describing the behavior of metals with isotropic hardening,

$$\phi({}^{t}f) = \left[\left(\frac{1}{2}\ {}^{t}s_{\alpha\beta}\ {}^{t}s_{\alpha\beta} \right)^{\frac{1}{2}} - \frac{{}^{t}\sigma_y}{\sqrt{3}} \right]^{\delta} . \tag{5.162}$$

In the above equation the term between brackets is the von Mises yield function.

Using the definition of the second invariant of the deviatoric Cauchy stresses we get,

$$\left. \frac{\partial f}{\partial \sigma_{\alpha\beta}} \right|_{t} = \frac{1}{2\sqrt{{}^{t}J_2}}\ {}^{t}s_{\alpha\beta} \tag{5.163}$$

hence, using Eq.(5.160), we get

$${}^{t}d_{\alpha\beta}^{VP} = \frac{\gamma}{2\sqrt{{}^{t}J_2}}\ {}^{t}s_{\alpha\beta}\ \left\langle\ {}^{t}f^{\delta}\ \right\rangle . \tag{5.164}$$

The above equation indicates that with the selected yield function the resulting viscoplastic flow is incompressible; a result that matches the experimental observations performed on the viscoplastic flow of metals.

Using the definition of equivalent viscoplastic strain associated to the von Mises yield function, Eq. (5.73a), we have

$${}^{t}\dot{\bar{\varepsilon}}_{VP} = \frac{\gamma}{\sqrt{3}} \left\langle\ {}^{t}f^{\delta}\ \right\rangle . \tag{5.165}$$

Therefore, for ${}^{t}f \geq 0$

$$\left({}^{t}f\right)^{\delta} = \frac{\sqrt{3}}{\gamma}\ {}^{t}\dot{\bar{\varepsilon}}_{VP} . \tag{5.166}$$

Formulating, for a rigid-viscoplastic material model, the relation among deviatoric stresses and strains as,

$${}^{t}s_{\alpha\beta} = 2{}^{t}\mu\ {}^{t}d_{\alpha\beta}^{VP} \tag{5.167}$$

and using the above equations we get, for ${}^{t}f \geq 0$

$${}^{t}\mu = \frac{\frac{{}^{t}\sigma_y}{\sqrt{3}} + \left[\frac{\sqrt{3}}{\gamma}\ {}^{t}\dot{\bar{\varepsilon}}_{VP} \right]^{\frac{1}{\delta}}}{\sqrt{3}\ {}^{t}\dot{\bar{\varepsilon}}_{VP}} . \tag{5.168}$$

From Eqs. (5.167) and (5.168) we see that a rigid-viscoplastic material behaves as a non-Newtonian fluid. It comes as no surprise that the solid behaves in a "fluid way", since we have neglected the solid elastic behavior and therefore its memory; the material memory is the main difference between the behavior of solids and fluids.

In the limit, when $\gamma \to \infty$ Eq. (5.168) describes the behavior of a rigid-plastic material (inviscid), in this case,

$$^{t}\mu = \frac{^{t}\sigma_y}{3 \ ^{t}\dot{\overline{\varepsilon}}_{VP}} \cdot \tag{5.169}$$

Example 5.16. ◀◀◀◀◀

An important experimentally observed effect, that the viscoplastic material model explains, is the increase in the apparent yield stress of metals when the strain rate is increased (Malvern 1969) (strain-rate effect).

Let us assume a uniaxial test in a rigid-viscoplastic bar,

$$\sigma_{11} = \widehat{\sigma}$$
$$\sigma_{22} = \sigma_{33} = 0 \cdot$$

Therefore,

$$s_{11} = \frac{2}{3}\widehat{\sigma}$$
$$s_{22} = s_{33} = -\frac{1}{3}\widehat{\sigma} \cdot$$

Also, for the viscoplastic strain rates we can write,

$$d_{11}^{VP} = \dot{\varepsilon}$$
$$d_{22}^{VP} = d_{33}^{VP} = -\frac{1}{2}\dot{\varepsilon} \cdot$$

Hence, the equivalent viscoplastic strain rate is,

$$\dot{\overline{\varepsilon}}_{VP} = \dot{\varepsilon} \ .$$

Using Eqs. (5.167) and (5.168) together with the above we get,

$$\widehat{\sigma} = \sigma_y \ + \ \sqrt{3} \left(\frac{\sqrt{3}}{\gamma} \dot{\varepsilon} \right)^{1/\delta} \ .$$

In the above equation, σ_y is the bar yield stress obtained with a quasistatic test and $\widehat{\sigma}$ is the apparent yield stress obtained with a dynamic test.

When $\gamma \to \infty$(inviscid plasticity), the strain-rate effect vanishes.

Using other functions in Eq. (5.162) more complicated strain-rate dependences can be explained (Backofen 1972). ◀◀◀◀◀

In (Zienkiewicz, Jain & Oñate 1977) a finite element methodology, based on a *rigid-viscoplastic* constitutive relation was developed, for analyzing bulk metal-forming processes. This methodology known as the *flow formulation* has been widely used since then for analyzing many industrial processes (Dvorkin, Cavaliere & Goldschmit 2003, Cavaliere, Goldschmit & Dvorkin 2001a\2001b, Dvorkin 2001, Dvorkin, Cavaliere & Goldschmit 1995\1997\1998, Dvorkin & Petöcz 1993).

5.5 Newtonian fluids

We define as an ideal or Newtonian fluid flow a viscous and incompressible one.

The first property of a Newtonian fluid is the lack of memory: Newtonian fluids do not present an elastic behavior and they do not store elastic energy.

Regarding the incompressible behavior we can write the continuity equation, using the result of Example 4.4 as,

$$\underline{\nabla} \cdot \, {}^t\underline{\mathbf{v}} = 0 \cdot \tag{5.170}$$

It is important to remark that even though there are some fluids that can be considered as incompressible, most of the cases of interest in engineering practice are flows where Eq. (5.170) is valid even though the fluids are not necessarily incompressible in all situations (e.g. isothermal air flow at low Mach numbers) (Panton 1984).

The constitutive relation for the Newtonian fluids can be written in the spatial configuration as,

$${}^t\underline{\underline{\boldsymbol{\sigma}}} = \, {}^tp \, {}^t\underline{\underline{\mathbf{g}}} + 2\mu \, {}^t\underline{\underline{\mathbf{d}}} \, . \tag{5.171}$$

In the above equation, ${}^t\underline{\underline{\boldsymbol{\sigma}}}$ is the Cauchy stress tensor, ${}^tp$ is its first invariant also called the mechanical pressure, ${}^t\underline{\underline{\mathbf{d}}}$ is the strain-rate tensor and μ is the fluid viscosity that we assume to be constant (it is usually called "molecular viscosity").

Note that for an incompressible flow ${}^td_{ii} = 0$ and therefore ${}^t\underline{\underline{\mathbf{d}}} = \, {}^t\underline{\underline{\mathbf{d}}}^D$.

Taking into account the incompressibility constraint in Eq. (5.170), it is important to realize that the pressure cannot not be associated to its energy conjugate: the volume strain rate, because it is zero; hence, the pressure will have to be determined from the equilibrium equations on the fluid-flow domain boundaries. Therefore it is not possible to solve an incompressible fluid flow in which all the boundary conditions are imposed velocities, at least at one boundary point we need to prescribe the tractions acting on it.

Many industrially important fluids, like polymers, do not obey Newton's constitutive equation. They are generally called non-Newtonian fluids. When

bulk metal forming processes are described neglecting the material elastic behavior (i.e. neglecting the material memory) the resulting constitutive equation is usually a non-Newtonian one (Zienkiewicz, Jain & Oñate 1977).

5.5.1 The no-slip condition

When solving a fluid flow usually two kinematic assumptions are made:

- At the interface between the fluid and the surrounding solid walls the velocity of the fluid normal to the walls is zero.
- At the interface between the fluid and the surrounding solid walls the velocity of the fluid tangential to the walls is zero.

The first of the above assumptions is quite obvious when referring to non-porous walls: the fluid cannot penetrate the walls.

The second of the above assumptions is not so obvious and, as a matter of fact, it has been historically the subject of much controversy; our faith in it is only pragmatic: it seems to work (Panton 1984).

6

Variational methods

In this chapter we will assume that the reader is familiar with the fundamentals of variational calculus. The topic can be studied from a number of references, among them (Fung 1965, Lanczos 1986, Segel 1987, Fung & Tong 2001).

The most natural way for starting the presentation of the theory of mechanics is by accepting the Principle of Momentum Conservation as a law of Nature and then stepping forward to demonstrate the Principle of Virtual Work as a consequence of the momentum conservation; this is perhaps the most direct way for developing the mechanical concepts because the Principle of Momentum Conservation is quite intuitive to the reader with a background in basic mechanics.

An alternative route for developing the theory of mechanics is by accepting the Principle of Virtual Work as a law of Nature and then stepping forward to demonstrate the Principle of Momentum Conservation. This route is perhaps not as intuitive as the first one but equally valid from a formal point of view.

However, more important than deciding which formulation is aesthetically more rewarding, an important fact for the scientist or engineer interested in solving advanced problems in mechanics is that the Principle of Virtual Work, and the other variational methods that can be derived from it, are the bases for the development of approximate solutions to problems for which analytical solutions cannot be found (Washizu 1982, Fung & Tong 2001).

6.1 The Principle of Virtual Work

We have represented in Fig. 6.1 the spatial configuration of a continuum body ${}^t\mathcal{B}$; its external surface ${}^tS$ can be subdivided into:
${}^tS_u$: on this surface the displacements are prescribed as boundary conditions,
${}^tS_\sigma$: on this surface the external loads are prescribed as boundary conditions.

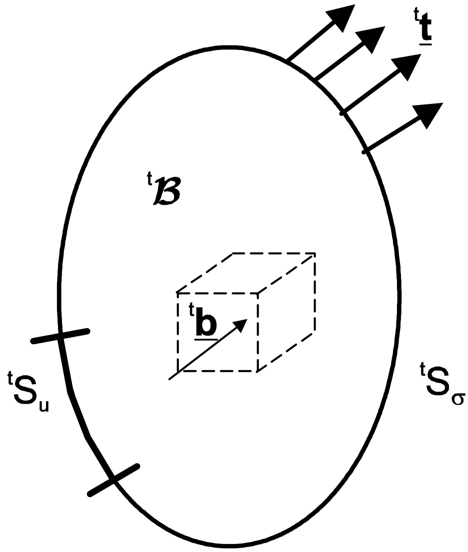


Fig. 6.1. Spatial configuration of a continuum

It is important to realize that a given point can pertain to ${}^tS_u$ in one direction and to ${}^tS_\sigma$ in another direction, *but* at one point, the displacement and the external load corresponding to the same direction cannot be simultaneously specified. Taking this into account we realize that the surfaces ${}^tS$, ${}^tS_u$ and ${}^tS_\sigma$ have to be defined as the addition of the surfaces corresponding to each of the three space directions. Also,

$$\begin{aligned} {}^tS &= {}^tS_u \cup {}^tS_\sigma \ , \\ {}^tS_u \cap {}^tS_\sigma &= \varnothing \ . \end{aligned}$$

The external forces acting on the body ${}^t\mathcal{B}$ in the spatial configuration are:
${}^t\underline{\mathbf{t}}$: external loads per unit surface acting on ${}^tS_\sigma$,
${}^t\underline{\mathbf{b}}$: external loads per unit mass.

We refer the continuum body to a spatial Cartesian coordinate system $\{{}^tz^\alpha\}$.

To each point in the $t-$configuration, which is an equilibrium configuration of the continuum body, we can associate a displacement vector ${}^t\underline{\mathbf{u}}$.

We can also define an *admissible displacements field* as (Fung 1965),

$$ {}^t\underline{\widetilde{\mathbf{u}}}({}^tz^\alpha) = {}^t\underline{\mathbf{u}}({}^tz^\alpha) + \delta{}^t\underline{\mathbf{u}}({}^tz^\alpha) \ . \tag{6.1}$$

In the above $\delta{}^t\underline{\mathbf{u}}$ is the *variation* of the displacements field, called the *virtual displacements*. The virtual displacements have to satisfy the boundary condition $\delta{}^t\underline{\mathbf{u}} \equiv \underline{0}$ on ${}^tS_u$ and they are arbitrary on ${}^tS_\sigma$, (Fung 1965).

Assuming that when the continuum evolves from ${}^t\underline{\mathbf{u}}$ to ${}^t\underline{\widetilde{\mathbf{u}}}$, the external loads remain constant, the work performed by them is the *virtual work of the external loads* ($\delta^t W_{ext}$),

$$\delta^t W_{ext} = \int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}} \; {}^t\rho \; {}^t\mathrm{d}V \; + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}} \; {}^t\mathrm{d}S \; . \tag{6.2}$$

Using in the above Eq.(3.7), which is an equilibrium equation for the particles on the body surface, we get

$$\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}} \; {}^t\mathrm{d}S = \int_{{}^tS_\sigma} \left({}^t\sigma_{\alpha\beta}\; \delta^t u_\beta\right) \; {}^t n_\alpha \; {}^t\mathrm{d}S \tag{6.3a}$$

$$= \int_{{}^tS} \left({}^t\sigma_{\alpha\beta}\; \delta^t u_\beta\right) \; {}^t n_\alpha \; {}^t\mathrm{d}S \tag{6.3b}$$

$$= \int_{{}^tV} \left({}^t\sigma_{\alpha\beta}\; \delta^t u_\beta\right)_{,\alpha} \; {}^t\mathrm{d}V \; . \tag{6.3c}$$

In the above, for deriving the last line we have used Gauss' theorem. Hence,

$$\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}} \; {}^t\mathrm{d}S = \int_{{}^tV} {}^t\sigma_{\alpha\beta,\alpha}\; \delta^t u_\beta \; {}^t\mathrm{d}V \; + \int_{{}^tV} {}^t\sigma_{\alpha\beta}\; \delta({}^t u_{\beta,\alpha}) \; {}^t\mathrm{d}V \; . \tag{6.4}$$

In the last integral we have used the equality (Fung 1965)

$$\delta\left(\frac{\partial^t u_\beta}{\partial^t z_\alpha}\right) = \frac{\partial}{\partial^t z_\alpha}\left(\delta^t u_\beta\right) \; . \tag{6.5}$$

Using in Eq. (6.4) the momentum conservation equation (Eq.(4.27b)),

$$\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}} \; {}^t\mathrm{d}S = -\int_{{}^tV} {}^t b_\alpha \; \delta^t u_\alpha \; {}^t\rho \; {}^t\mathrm{d}V + \int_{{}^tV} {}^t\sigma_{\alpha\beta}\; \delta\left({}^t u_{\beta,\alpha}\right) \; {}^t\mathrm{d}V \; . \tag{6.6}$$

In the derivation of the above equation we have assumed in Eq. (4.27b) that $\frac{D^t\mathbf{v}}{Dt} = 0$; however, dynamic problems can also be considered by including the inertia forces among the external loads per unit mass (Crandall 1956).

It is easy to show that,

$${}^t\sigma_{\alpha\beta}\; \delta^t u_{\beta,\alpha} = \; {}^t\sigma_{\alpha\beta}\; \delta^t\varepsilon_{\alpha\beta} \tag{6.7a}$$

$$\delta^t\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial\left(\delta^t u_\alpha\right)}{\partial^t z_\beta} + \frac{\partial\left(\delta^t u_\beta\right)}{\partial^t z_\alpha}\right] \; . \tag{6.7b}$$

In the above equations the terms $\delta^t\varepsilon_{\alpha\beta}$ are the infinitesimal strain components developed by the virtual displacements; hence, we refer to them as virtual strain components.

Note that the actual strains in the t-configuration are arbitrary, only the *virtual strains* are infinitesimal.

Replacing with Eq. (6.7a) in Eq. (6.6),

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\, {}^t\rho\, {}^t\mathrm{d}V + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\, {}^t\mathrm{d}S = \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} : \delta^t\underline{\underline{\boldsymbol{\varepsilon}}}\, {}^t\mathrm{d}V \,. \tag{6.8}$$

The above equation is the mathematical statement of the Principle of Virtual Work and it states that for a continuum body in equilibrium, the virtual work of the external loads equals the virtual work of the stresses.

Notes:

- No assumption was made on the material, i.e. on its stress - strain relation; hence, the Principle of Virtual Work holds for any constitutive relation.
- No assumption was made on the actual strains in the spatial configuration; hence, the Principle of Virtual Work holds for finite or infinitesimal strains.
- No assumption was made on the external loads; hence, the Principle of Virtual Work holds for conservative and nonconservative loads (Crandall 1956).
- The integrals in Eq. (6.8) are calculated on the spatial configuration of the body.
- The Principle of Virtual Work was derived from momentum conservation **and not** from energy conservation.

As we see the Principle of Virtual Work is a very general statement, holding for any type of nonlinearities that may be present in the spatial configuration (material and geometrical nonlinearities).

It is also possible to go through the inverse route, that is to say, starting from the Principle of Virtual Work to demonstrate the equations of momentum conservation. For this demonstration we refer the reader to (Fung 1965, Fung & Tong 2001).

6.2 The Principle of Virtual Work in geometrically nonlinear problems

In Eq. (6.8) the integrals are calculated in the spatial configuration of the continuum, which is normally one of the problem unknowns; however, for geometrically linear problems (${}^tu_{\alpha,\beta} << 1$) the difference between the spatial configuration and the unloaded configuration, normally the reference configuration, is negligible; hence

$${}^tV \cong {}^\circ V \tag{6.9a}$$

$${}^tS \cong {}^\circ S \,. \tag{6.9b}$$

In the case of geometrically nonlinear problems it is convenient to calculate the integrals in Eq. (6.8) using the reference configuration.

The coordinates ${}^t z_\alpha$ remain constant during the virtural displacement; hence,

$$\frac{\mathrm{d}\delta^t \varepsilon_{\alpha\beta}}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\partial \delta^t v_\alpha}{\partial^t z_\beta} + \frac{\partial \delta^t v_\beta}{\partial^t z_\alpha}\right) \tag{6.10}$$

where $\delta^t \underline{\mathbf{v}}$ is the virtual velocity vector.

Equation (3.11) is valid for any velocity field, in particular when we use the virtual velocity field, we get using Eq. (6.10),

$$\int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} : \delta^t \underline{\underline{\boldsymbol{\varepsilon}}} \; {}^t\mathrm{d}V = \int_{{}^\circ V} {}^t\underline{\underline{\boldsymbol{\tau}}} : \delta^t \underline{\underline{\boldsymbol{\varepsilon}}} \; {}^\circ\mathrm{d}V \; . \tag{6.11}$$

Replacing in Eq. (6.8),

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t \underline{\mathbf{u}} \; {}^t\rho \; {}^t\mathrm{d}V \; + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t \underline{\mathbf{u}} \; {}^t\mathrm{d}S = \int_{{}^\circ V} {}^t\underline{\underline{\boldsymbol{\tau}}} : \delta^t \underline{\underline{\boldsymbol{\varepsilon}}} \; {}^\circ\mathrm{d}V \; . \tag{6.12}$$

Also, using another pair of energy conjugate measures,

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t \underline{\mathbf{u}} \; {}^t\rho \; {}^t\mathrm{d}V \; + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t \underline{\mathbf{u}} \; {}^t\mathrm{d}S = \int_{{}^\circ V} {}^t_o\underline{\underline{\mathbf{S}}} : \boldsymbol{\delta} \; {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} \; {}^\circ\mathrm{d}V \; . \tag{6.13}$$

In the above,
${}^t_o\underline{\underline{\mathbf{S}}}$: second Piola-Kirchhoff stress tensor,
${}^t_o\underline{\underline{\boldsymbol{\varepsilon}}}$: Green-Lagrange strain tensor.

We can define a general load, either a load per unit mass or per unit surface as:

$${}^t\underline{\mathbf{f}} = \; {}^l\lambda \; {}^m\underline{\boldsymbol{\varphi}} \; . \tag{6.14}$$

In the above, ${}^l\lambda$ is a scalar proportional to the load modulus and (Schweizerhof & Ramm 1984):

- $l = 0$ implies that the load modulus is a function of the reference coordinates *(body-attached loads)*;
- $l = t$ implies that the load modulus is a function of the spatial coordinates *(space-attached loads).*

The unitary vector ${}^m\underline{\boldsymbol{\varphi}}$ is a direction and (Schweizerhof & Ramm 1984):

- $m = 0$ implies a constant direction load;
- $m = t$ implies a follower load (the direction is a function of the body displacements).

Example 6.1. ◀◀◀◀◀

Buckling of a circular ring.

In (Brush & Almroth 1975) we find that the elastic buckling pressure acting on a circular ring depends on the type of load that we consider:

Load	*Buckling pressure*
Hydrostatic pressure loading	$p_{cr} = 3\frac{EI}{a^3}$
Centrally directed pressure loading	$p_{cr} = 4.5\frac{EI}{a^3}$

Both are cases of follower loads but, the description of the load as a function of the displacements is different. ◀◀◀◀◀

Using the mass conservation principle in Eq. (4.20d) we can write,

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\; {}^t\rho\; {}^t\mathrm{d}V = \int_{{}^\circ V} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\; {}^o\rho\; {}^\circ\mathrm{d}V\ . \tag{6.15}$$

At each point on the surface bounding the continua we can calculate,

$${}^t\mathrm{d}S = {}^tJ_S\; {}^\circ\mathrm{d}S\ , \tag{6.16}$$

hence,

$$\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\; {}^t\mathrm{d}S = \int_{{}^\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\; {}^tJ_S\; {}^\circ\mathrm{d}S\ . \tag{6.17}$$

Therefore, we can write the principle of Virtual Work calculating the integrals over the reference configuration as,

$$\int_{{}^\circ V} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\; {}^o\rho\; {}^\circ\mathrm{d}V + \int_{{}^\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\; {}^tJ_S\; {}^\circ\mathrm{d}S = \int_{{}^\circ V} {}^t_o\underline{\underline{\mathbf{S}}} : \boldsymbol{\delta}\, {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}}\; {}^\circ\mathrm{d}V. \tag{6.18}$$

Example 6.2. ◀◀◀◀◀

Let us consider the work of the external loads per unit surface of the spatial configuration for the case of a typical follower load: the hydrostatic fluid pressure. In this case,

$${}^t\underline{\mathbf{t}} = {}^tp\; {}^t\underline{\mathbf{n}}$$

where ${}^t\underline{\mathbf{n}}$ is the surface external normal. For this case,

$$\delta^tW^S_{ext} = \int_{{}^tS_\sigma} {}^tp\; \delta^t\underline{\mathbf{u}} \cdot {}^t\underline{\mathbf{n}}\; {}^t\mathrm{d}S\ .$$

Using Nanson's formula (Example 4.9) we get,

$$\delta^t W_{ext}^S = \int_{^\circ S_\sigma} {}^t p \, \delta^t \underline{\mathbf{u}} \cdot {}^t J \, {}^\circ \underline{\mathbf{n}} \cdot {}^t_o \underline{\underline{\mathbf{X}}}^{-1} \, {}^\circ \mathrm{d}S \ .$$

For a case with infinitesimal strains,

$$\begin{aligned} {}^t J &\approx 1 \\ {}^t_o \underline{\underline{\mathbf{X}}}^{-1} &\approx {}^t_o \underline{\underline{\mathbf{R}}}^T \end{aligned}$$

and therefore,

$$\delta W_{ext}^S = \int_{^\circ S_\sigma} \left[{}^t_o \underline{\underline{\mathbf{R}}}^T \cdot \delta^t \underline{\mathbf{u}} \right] \cdot \left[{}^t p \, {}^\circ \underline{\mathbf{n}} \, \right] \, {}^\circ \mathrm{d}S \ .$$

In the above equation, the first bracket inside the integral is the displacements variation rotated back from the spatial configuration to the material one; and the second bracket is the load per unit surface, but in the reference configuration.
Hence, it is very important to realize that for the case of fluid-pressure loads and infinitesimal strains we can easily calculate the external surface loads virtual work in the reference configuration. ◀◀◀◀◀

Using other alternative energy conjugate stress/strain measures,

$$\int_{^\circ V} {}^t \underline{\mathbf{b}} \cdot \delta^t \underline{\mathbf{u}} \, {}^o \rho \, {}^\circ \mathrm{d}V + \int_{^\circ S_\sigma} {}^t \underline{\mathbf{t}} \cdot \delta^t \underline{\mathbf{u}} \, {}^t J_S \, {}^\circ \mathrm{d}S = \int_{^\circ V} {}^t_o \underline{\underline{\mathbf{P}}}^T : \boldsymbol{\delta} \, {}^t_o \underline{\underline{\mathbf{X}}} \, {}^\circ \mathrm{d}V \ , \quad (6.19)$$

and, for an isotropic constitutive relation

$$\int_{^\circ V} {}^t \underline{\mathbf{b}} \cdot \delta^t \underline{\mathbf{u}} \, {}^o \rho \, {}^\circ \mathrm{d}V + \int_{^\circ S_\sigma} {}^t \underline{\mathbf{t}} \cdot \delta^t \underline{\mathbf{u}} \, {}^t J_S \, {}^\circ \mathrm{d}S = \int_{^\circ V} {}^t \underline{\underline{\boldsymbol{\Gamma}}} : \boldsymbol{\delta} \, {}^t_o \underline{\underline{\mathbf{H}}} \, {}^\circ \mathrm{d}V \ . \quad (6.20)$$

6.2.1 Incremental Formulations

We have represented in Fig. 6.2 a typical Lagrangian analysis where the configuration at $t = 0$ is known and the configuration at $t = t_1$ is sought.

Normally, we perform an incremental analysis; that is to say, we determine the sequence of equilibrium configurations at $t = 0, ..., t, t + \Delta t, \ ...t_1$.

In this incremental analysis, the basic link to be analyzed is the generic step $t \rightarrow t + \Delta t$; that is to say, knowing the equilibrium configuration at t we seek the one at $t + \Delta t$. Of course, once this generic step can be solved, then the complete incremental analysis can be performed.

In what follows, to describe the step $t \rightarrow t + \Delta t$ we follow the presentation in (Bathe 1996).

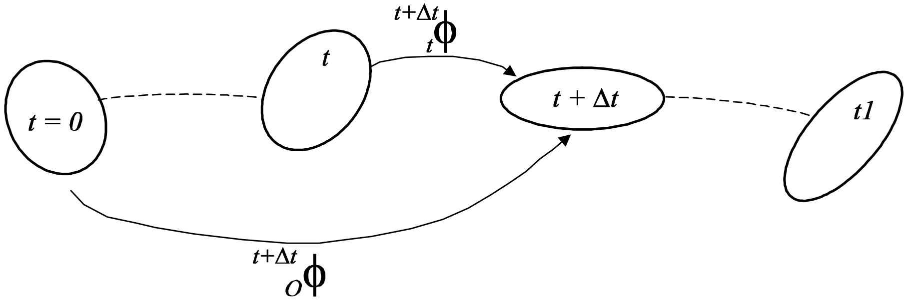


Fig. 6.2. Lagragian incremental analysis

First, we have to recognize that for describing the $t + \Delta t$-configuration we can use as a reference configuration either the one at $t = 0$ or any of the intermediate ones, already known. In what follows we will specifically analyze two particular cases:

- The *total Lagrangian formulation,* where we use as the reference configuration the one at $t = 0$ (usually the undeformed configuration)
- The *updated Lagrangian formulation,* where we use as the reference configuration the previous one (t).

The total Lagrangian formulation
Using the principle of Virtual Work to define the $t + \Delta t$ equilibrium configuration, we write:

$$\int_{^\circ V} {}^{t+\Delta t}_{o}S^{IJ}\ \delta^{t+\Delta t}_{o}\varepsilon_{IJ}\ ^\circ\mathrm{d}V = \delta^{t+\Delta t}W_{ext}\ . \qquad (6.21)$$

In the above equation,
${}^{t+\Delta t}_{o}S^{IJ}$: components of the second Piola-Kirchhoff stress tensor, corresponding to the $t + \Delta t$ configuration and referred to the configuration at $t = 0$.
${}^{t+\Delta t}_{o}\varepsilon_{IJ}$: components of the Green-Lagrange strain tensor, corresponding to the $t + \Delta t$ configuration and referred to the configuration at $t = 0$.

A general curvilinear coordinate system $\left\{^\circ x^I, I = 1, 2, 3\right\}$ is used in the reference configuration, with covariant base vectors $^\circ\underline{\mathbf{g}}_L$ $(L = 1, 2, 3)$.

The volume of the reference configuration is $^\circ V$ and the virtual work of the external loads acting at time $t + \Delta t$ is $\delta^{t+\Delta t}W_{ext}$ (see the previous section for a discussion on the calculation of this term).

We can write,

$${}^{t+\Delta t}_{o}S^{IJ} = {}^{t}_{o}S^{IJ} + {}_{o}S^{IJ} \qquad (6.22)$$

where ${}_{o}S^{IJ}$ are the components of the incremental second Piola-Kirchhoff stress tensor. It is important to recognize that the three tensors in Eq. (6.22)

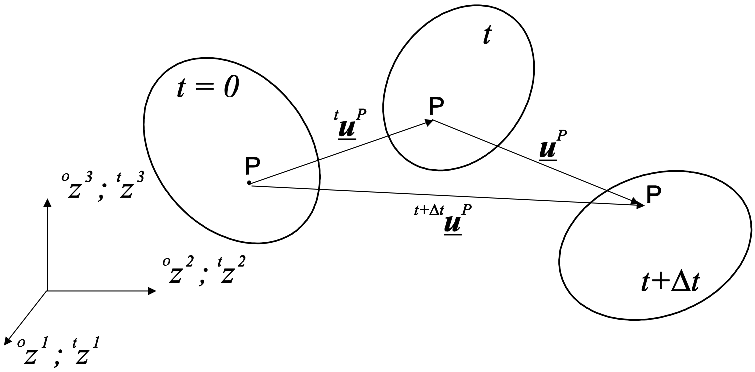


Fig. 6.3. Total Lagrangian formulation

are referred to the same reference configuration ($t = 0$). For the Green-Lagrange strain tensor we can also write an incremental equation,

$$ {}^{t+\Delta t}_{\;\;o}\varepsilon_{IJ} = {}^{t}_{o}\varepsilon_{IJ} + {}_{o}\varepsilon_{IJ} \;, \tag{6.23}$$

again, in the above equation the three tensors are referred to the same reference configuration ($t = 0$).

Replacing with Eqs. (6.22) and (6.23) in Eq. (6.21) and taking into account that for the step that we are investigating, ${}^{t}_{o}\varepsilon_{IJ}$ is data and therefore, $\delta^{t}_{o}\varepsilon_{IJ} = 0$, we get

$$\int_{{}^{\circ}V} \left({}^{t}_{o}S^{IJ} + {}_{o}S^{IJ}\right) \delta_{o}\varepsilon_{IJ} \; {}^{\circ}\mathrm{d}V = \delta^{t+\Delta t}W_{ext} \;. \tag{6.24}$$

We write an incremental constitutive equation of the form,

$$ {}_{o}S^{IJ} = {}_{o}C^{IJKL} \; {}_{o}\varepsilon_{KL} \tag{6.25}$$

and get,

$$\int_{{}^{\circ}V} \left({}^{t}_{o}S^{IJ} + {}_{o}C^{IJKL} \; {}_{o}\varepsilon_{KL}\right) \delta_{o}\varepsilon_{IJ} \; {}^{\circ}\mathrm{d}V = \delta^{t+\Delta t}W_{ext} \;. \tag{6.26}$$

Refering the problem to a fixed Cartesian system, we can write for a generic particle P, as shown in Fig. 6.3:

$$ {}^{t}\underline{\mathbf{u}}^{P} = {}^{t}\underline{\mathbf{x}}^{P} - {}^{\circ}\underline{\mathbf{x}}^{P} \tag{6.27a}$$

$$ \underline{\mathbf{u}}^{P} = {}^{t+\Delta t}\underline{\mathbf{u}}^{P} - {}^{t}\underline{\mathbf{u}}^{P} \;. \tag{6.27b}$$

We can show that (Bathe 1996),

$$ {}_o\varepsilon_{\alpha\beta} = \frac{1}{2}\left({}_ou_{\alpha,\beta} + {}_ou_{\beta,\alpha} + {}^t_ou_{\gamma,\alpha}\ {}_ou_{\gamma,\beta} + {}^t_ou_{\gamma,\beta}\ {}_ou_{\gamma,\alpha} + {}_ou_{\gamma,\alpha}\ {}_ou_{\gamma,\beta}\right) \ . \tag{6.28} $$

In the above equation, ${}_ou_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial^\circ z^\beta}$ and ${}^t_ou_{\alpha,\beta} = \frac{\partial^t u_\alpha}{\partial^\circ z^\beta}$.

Hence, it is possible to rewrite Eq. (6.28) as,

$$ {}_o\varepsilon_{\alpha\beta} = {}_oe_{\alpha\beta} + {}_o\eta_{\alpha\beta} \tag{6.29a} $$

$$ {}_oe_{\alpha\beta} = \frac{1}{2}\left({}_ou_{\alpha,\beta} + {}_ou_{\beta,\alpha} + {}^t_ou_{\gamma,\alpha}\ {}_ou_{\gamma,\beta} + {}^t_ou_{\gamma,\beta}\ {}_ou_{\gamma,\alpha}\right) \tag{6.29b} $$

$$ {}_o\eta_{\alpha\beta} = \frac{1}{2}\ {}_ou_{\gamma,\alpha}\ {}_ou_{\gamma,\beta} \ . \tag{6.29c} $$

The term in Eq. (6.29b) is linear in the unknown incremental displacements, $\underline{\mathbf{u}}$, while the term in Eq. (6.29c) is nonlinear.

We introduce Eq. (6.29a) in Eq. (6.26) and obtain,

$$ \int_{^\circ V} \left[{}^t_oS_{\alpha\beta} + {}_oC_{\alpha\beta\gamma\delta}\ ({}_oe_{\gamma\delta} + {}_o\eta_{\gamma\delta})\right] \delta({}_oe_{\alpha\beta} + {}_o\eta_{\alpha\beta})\ {}^\circ\mathrm{d}V = \delta^{t+\Delta t}W_{ext} \ . \tag{6.30} $$

The above is the momentum balance equation at time $t + \Delta t$; which is a nonlinear equation in the incremental displacement vector. In order to solve it we use an iterative technique, in which the first step is the linearization of Eq. (6.30) (Bathe 1996). Keeping only up to the linear terms in $\underline{\mathbf{u}}$, we obtain the *linearized momentum balance equation*:

$$ \begin{aligned} &\int_{^\circ V} \left[{}_oC_{\alpha\beta\gamma\delta}\ {}_oe_{\gamma\delta}\ \delta_oe_{\alpha\beta} + {}^t_oS_{\alpha\beta}\ \delta_o\eta_{\alpha\beta}\right] {}^\circ\mathrm{d}V \\ &= \delta^{t+\Delta t}W_{ext} - \int_{^\circ V} {}^t_oS_{\alpha\beta}\ \delta_oe_{\alpha\beta}\ {}^\circ\mathrm{d}V \ . \end{aligned} \tag{6.31} $$

The updated Lagrangian formulation

Using the principle of Virtual Work to define the $t + \Delta t$ equilibrium configuration, we write:

$$ \int_{^tV} {}^{t+\Delta t}_{\ \ t}S^{IJ}\ \delta^{t+\Delta t}_{\ \ t}\varepsilon_{IJ}\ {}^t\mathrm{d}V = \delta^{t+\Delta t}W_{ext} \ . \tag{6.32} $$

In the above equation,

${}^{t+\Delta t}_{\ \ t}S^{IJ}$: components of the second Piola-Kirchhoff stress tensor, corresponding to the $t + \Delta t$ configuration and referred to the configuration at t.

${}^{t+\Delta t}_{\ \ t}\varepsilon_{IJ}$: components of the Green-Lagrange strain tensor, corresponding to the $t + \Delta t$ configuration and referred to the configuration at t.

We can write,

$$ {}^{t+\Delta t}_{\ \ t}S^{IJ} = {}^t_tS^{IJ} + {}_tS^{IJ} \tag{6.33} $$

where ${}^{t}_{t}S^{IJ} = {}^{t}\sigma^{IJ}$ and ${}_{t}S^{IJ}$ are the components of the incremental second Piola-Kirchhoff stress tensor; it is important to recognize that the three tensors in Eq. (6.33) are referred to the spatial configuration at time t.

Also,

$$ {}^{t+\Delta t}_{t}\varepsilon_{IJ} = {}_{t}\varepsilon_{IJ} \quad . \tag{6.34} $$

because ${}^{t}_{t}\varepsilon_{IJ} = 0$.

Replacing with Eqs. (6.33) and (6.34) in Eq. (6.32), we get

$$ \int_{{}^{t}V} \left({}^{t}\sigma^{IJ} + {}_{t}S^{IJ} \right) \delta \left({}_{t}\varepsilon_{IJ} \right) {}^{t}\mathrm{d}V = \delta^{t+\Delta t} W_{ext} \, , \tag{6.35} $$

we can write an incremental constitutive equation referred to the $t-$configuration,

$$ {}_{t}S^{IJ} = {}_{t}C^{IJKL} \; {}_{t}\varepsilon_{KL} \, , \tag{6.36} $$

and get,

$$ \int_{{}^{t}V} \left({}^{t}\sigma^{IJ} + {}_{t}C^{IJKL} \; {}_{t}\varepsilon_{KL} \right) \delta \left({}_{t}\varepsilon_{IJ} \right) {}^{t}\mathrm{d}V = \delta^{t+\Delta t} W_{ext} \, . \tag{6.37} $$

In a fixed Cartesian system we can show that (Bathe 1996),

$$ {}_{t}\varepsilon_{\alpha\beta} = \frac{1}{2} \left({}_{t}u_{\alpha,\beta} + {}_{t}u_{\beta,\alpha} + {}_{t}u_{\gamma,\alpha} \; {}_{t}u_{\gamma,\beta} \right) \tag{6.38} $$

where ${}_{t}u_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial^{t} z_\beta}$.

We can decompose the strain increment into a linear and a nonlinear increment in the unknown incremental displacement; that is to say,

$$ \begin{aligned} {}_{t}\varepsilon_{\alpha\beta} &= {}_{t}e_{\alpha\beta} + {}_{t}\eta_{\alpha\beta} \\ {}_{t}e_{\alpha\beta} &= \frac{1}{2} \left({}_{t}u_{\alpha,\beta} + {}_{t}u_{\beta,\alpha} \right) \\ {}_{t}\eta_{\alpha\beta} &= \frac{1}{2} \left({}_{t}u_{\gamma,\alpha} \; {}_{t}u_{\gamma,\beta} \right) \quad . \end{aligned} \tag{6.39} $$

Hence we can write Eq. (6.37) as,

$$ \int_{{}^{t}V} \left[{}^{t}\sigma_{\alpha\beta} + {}_{t}C_{\alpha\beta\gamma\delta} \left({}_{t}e_{\gamma\delta} + {}_{t}\eta_{\gamma\delta} \right) \right] \delta \left({}_{t}e_{\alpha\beta} + {}_{t}\eta_{\alpha\beta} \right) {}^{t}\mathrm{d}V = \delta^{t+\Delta t} W_{ext} \, . \tag{6.40} $$

The above is the momentum balance equation at time $t + \Delta t$; which is a nonlinear equation in the incremental displacement vector. Proceeding in the same way as in the total Lagrangian formulation we obtain the *linearized momentum balance equation* (Bathe 1996):

$$ \begin{aligned} &\int_{{}^{t}V} {}_{t}C_{\alpha\beta\gamma\delta} \; {}_{t}e_{\gamma\delta} \; \delta_{t}e_{\alpha\beta} \; {}^{t}\mathrm{d}V + \int_{{}^{t}V} {}^{t}\sigma_{\alpha\beta} \; \delta_{t}\eta_{\alpha\beta} \; {}^{t}\mathrm{d}V \\ &\quad = \delta^{t+\Delta t} W_{ext} - \int_{{}^{t}V} {}^{t}\sigma_{\alpha\beta} \; \delta_{t}e_{\alpha\beta} \; {}^{t}\mathrm{d}V \, . \end{aligned} \tag{6.41} $$

It is easy to show that

$$
{}_{o}S^{IJ} = \frac{{}^{o}\rho}{{}^{t}\rho}\, {}_{t}S^{ij} \;\left({}^{t}_{o}X^{-1}\right)^{I}_{\;i} \;\left({}^{t}_{o}X^{-1}\right)^{J}_{\;j} \tag{6.42}
$$

$$
{}_{o}\varepsilon_{IJ} = {}_{t}\varepsilon_{ij}\; {}^{t}_{o}X^{i}_{\;I}\; {}^{t}_{o}X^{j}_{\;J} \tag{6.43}
$$

and therefore if the same material is considered in both formulations the incremental constitutive tensors should be related,

$$
{}_{\circ}C^{IJKL} = \frac{{}^{o}\rho}{{}^{t}\rho}\, {}_{t}C^{mnpq} \;\left({}^{t}_{o}X^{-1}\right)^{I}_{\;m} \;\left({}^{t}_{o}X^{-1}\right)^{J}_{\;n} \;\left({}^{t}_{o}X^{-1}\right)^{K}_{\;p} \;\left({}^{t}_{o}X^{-1}\right)^{L}_{\;q} \;. \tag{6.44}
$$

Any problem can be alternatively solved using either the total or the updated Lagrangian formulations and the results should be identical (Bathe 1996).

For solving finite-strain elastoplastic problems, in Sect. 5.2.6 we introduced an adhoc incremental formulation, the *total Lagrangian-Hencky formulation.*

6.3 The Principle of Virtual Power

There are formulations where the primary unknowns are the material velocities rather than the material displacements (e.g. fluid problems, metal-forming Eulerian (Dvorkin, Cavaliere & Goldschmit 1995, Dvorkin & Petöcz 1993) or ALE formulations (Belytschko, Liu & Moran 2000), etc.). For these cases the momentum conservation leads to,

$$
\int_{{}^{t}V} {}^{t}\underline{\mathbf{b}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{t}\rho\; {}^{t}\mathrm{d}V \;+\; \int_{{}^{t}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{t}\mathrm{d}S = \int_{{}^{t}V} {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : \delta^{t}\underline{\underline{\mathbf{d}}}\; {}^{t}\mathrm{d}V \quad . \tag{6.45}
$$

In the above equation ${}^{t}\underline{\mathbf{v}}$ is the material velocity at a point and ${}^{t}\underline{\underline{\mathbf{d}}}$ is the strain-rate tensor.

Of course, we can use, for formulating the Principle of Virtual Power, other energy conjugated stress/strain rate measures, for example:

$$
\int_{{}^{\circ}V} {}^{t}\underline{\mathbf{b}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{o}\rho\; {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{t}J_{S}\; {}^{\circ}\mathrm{d}S = \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\tau}}} : \delta^{t}\underline{\underline{\mathbf{d}}}\; {}^{\circ}\mathrm{d}V, \tag{6.46a}
$$

$$
\int_{{}^{\circ}V} {}^{t}\underline{\mathbf{b}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{o}\rho\; {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}}^{t} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{t}J_{S}\; {}^{\circ}\mathrm{d}S = \int_{{}^{\circ}V} {}^{t}_{o}\underline{\underline{\mathbf{S}}} : \delta^{t}_{o}\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}\; {}^{\circ}\mathrm{d}V, \tag{6.46b}
$$

$$
\int_{{}^{\circ}V} {}^{t}\underline{\mathbf{b}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{o}\rho\; {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{t}J_{S}\; {}^{\circ}\mathrm{d}S = \int_{{}^{\circ}V} {}^{t}_{o}\underline{\underline{\mathbf{P}}}^{T} : \delta^{t}_{o}\dot{\underline{\underline{\mathbf{X}}}}\; {}^{\circ}\mathrm{d}V, \tag{6.46c}
$$

$$
\int_{{}^{\circ}V} {}^{t}\underline{\mathbf{b}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{o}\rho\; {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{v}}\; {}^{t}J_{S}\; {}^{\circ}\mathrm{d}S = \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\Gamma}}} : \delta^{t}_{o}\dot{\underline{\underline{\mathbf{H}}}}\; {}^{\circ}\mathrm{d}V, \tag{6.46d}
$$

the last one only being valid for isotropic constitutive relations.

6.4 The Principle of Stationary Potential Energy

As we remarked above, the Principle of Virtual Work can be used for any material constitutive relation, for any type of loading and for any nonlinearity in the case to be analyzed.

In the present section we will specialize the Principle of Virtual Work for:

- Hyperelastic materials.
- Conservative external loads.

For a hyperelastic material we have seen in Chap. 5 (Eq. (5.3d) that,

$$ {}_{o}^{t}S^{IJ} = {}^{o}\rho \, \frac{\partial \, {}^{t}\mathtt{U}({}_{o}^{t}\underline{\underline{\varepsilon}})}{\partial \, {}_{o}^{t}\varepsilon_{IJ}} \ . \tag{6.47} $$

The external *conservative loads* are the external loads that can be derived from a potential. Hence, a load field is said to be conservative in a region if the net work done around any closed path in that region is zero (Crandall 1956).

A typical conservative load system can be represented as,

$$ {}^{t}\underline{\mathbf{f}} = {}^{\circ}\lambda \, {}^{\circ}\underline{\boldsymbol{\varphi}} \ . \tag{6.48} $$

Following the definitions introduced above, the load system in Eq. (6.48) is a body attached load system with constant direction.

For conservative loads per unit mass, we write

$$ {}^{t}\underline{\mathbf{b}} = -\frac{\partial^{t}G\left({}^{t}\underline{\mathbf{u}}\right)}{\partial \underline{\mathbf{u}}} \tag{6.49} $$

and for conservative surface loads

$$ {}^{t}\underline{\mathbf{t}} = -\frac{\partial^{t}g\left({}^{t}\underline{\mathbf{u}}\right)}{\partial \underline{\mathbf{u}}} \ . \tag{6.50} $$

Note that the above-defined surface loads are defined as loads per unit reference surface; therefore, its resultant at time t is $\int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \ {}^{\circ}\mathrm{d}S$.

We now define a functional of the function ${}^{t}\underline{\mathbf{u}}$ called the *potential energy* functional:

$$ {}_{o}^{t}\Pi = \int_{{}^{\circ}V} {}^{\circ}\rho \ \left({}^{t}\mathtt{U} \ + \ {}^{t}G\right) \ {}^{\circ}\mathrm{d}V \ + \ \int_{{}^{\circ}S_{\sigma}} {}^{t}g \ \ {}^{\circ}\mathrm{d}S \tag{6.51} $$

Therefore,

$$\delta^t_\circ \varPi = \int_{^\circ V} \left[{}^\circ\rho \frac{\partial^t \mathtt{U}}{\partial^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} : \boldsymbol{\delta}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} \; + \; {}^\circ\rho \frac{\partial^t G}{\partial \underline{\mathbf{u}}} \cdot \delta^t \underline{\mathbf{u}} \right] {}^\circ\mathrm{d}V + \int_{^\circ S_\sigma} \frac{\partial \, {}^t g}{\partial \underline{\mathbf{u}}} \cdot \delta^t \underline{\mathbf{u}} \, {}^\circ\mathrm{d}S \,. \tag{6.52}$$

In the above, $\delta^t \underline{\mathbf{u}}$ are admissible variations $\left(\delta^t \underline{\mathbf{u}} = \underline{\mathbf{0}} \text{ on } {}^\circ S_u \;\text{ see Fig. 6.1 }\right)$ and $\boldsymbol{\delta}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}$ is derived from the displacement variations. Therefore,

$$\delta^t_\circ \varPi = \int_{^\circ V} \left[{}^t_\circ \underline{\underline{\mathbf{S}}} : \boldsymbol{\delta}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} \; - \; {}^\circ\rho \, {}^t \underline{\mathbf{b}} \cdot \delta^t \underline{\mathbf{u}} \right] {}^\circ\mathrm{d}V \; - \; \int_{^\circ S_\sigma} {}^t \underline{\mathbf{t}} \cdot \delta^t \underline{\mathbf{u}} \, {}^\circ\mathrm{d}S \,. \tag{6.53}$$

Hence, for a hyperelastic material under a conservative load system, the principle of virtual work, in Eq. (6.18), can be written as

$$\delta^t_\circ \varPi = 0 \,. \tag{6.54}$$

The above equation states that when the $t-$configuration is in equilibrium the potential energy functional reaches a stationary value; i.e. it fulfills the necessary requirements for being an extreme (Fung 1965).

In what follows we show that in the case of infinitesimal strains the potential energy not only is stationary at the equilibrium configuration but it actually attains there a minimum.

Using the nomenclature introduced in Eq. (6.1) we write the potential energy functional for an admissible configuration close to the equilibrium one as

$$\begin{aligned} {}^t_o \varPi^{'} = & \int_{^\circ V} {}^\circ\rho \left[{}^t \mathtt{U}({}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} + \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}) \; - \; {}^t \underline{\mathbf{b}} \cdot \left({}^t \underline{\mathbf{u}} + \delta^t \underline{\mathbf{u}} \right) \right] {}^\circ\mathrm{d}V \\ & - \int_{^\circ S_\sigma} {}^t \underline{\mathbf{t}} \cdot \left({}^t \underline{\mathbf{u}} + \delta^t \underline{\mathbf{u}} \right) {}^\circ\mathrm{d}S \,. \end{aligned} \tag{6.55}$$

Using a Taylor expansion,

$${}^t \mathtt{U}({}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} + \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}) \; = \; {}^t \mathtt{U}({}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}) + \left. \frac{\partial^t \mathtt{U}}{\partial^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} \right|_{{}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} : \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} + \frac{1}{2} \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} : \left. \frac{\partial^{2\,t} \mathtt{U}}{\partial^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} \, \partial^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} \right|_{{}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} : \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} + \cdots \quad . \tag{6.56}$$

Hence,

$${}^t_o \varPi^{'} - {}^t_o \varPi = \delta^t_\circ \varPi + \int_{^\circ V} \frac{1}{2} \, {}^\circ\rho \; \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} : \left. \frac{\partial^{2\,t} \mathtt{U}}{\partial^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} \, \partial^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} \right|_{{}^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}}} : \delta^t_\circ \underline{\underline{\boldsymbol{\varepsilon}}} \, {}^\circ\mathrm{d}V + \cdots \,. \tag{6.57}$$

Since at equilibrium $\delta^t_\circ \varPi = 0$, the sign of the l.h.s. is the sign of the integrand on the r.h.s..

In the case of infinitesimal strains case we can assume that ${}^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}} \approx \underline{\underline{\mathbf{0}}}$ and we have ${}^t\mathtt{U}(\underline{\underline{\mathbf{0}}}) = 0$ (convention) and ${}^t_\circ\underline{\underline{\mathbf{S}}}\Big|_{\underline{\underline{\mathbf{0}}}} = {}^\circ\rho\frac{\partial^t\mathtt{U}}{\partial^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}}\Big|_{\underline{\underline{\mathbf{0}}}} = \underline{\underline{\mathbf{0}}}$; hence, from Eq. (6.56)

$$
{}^t\mathtt{U}(\delta^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\,\delta^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}} : \frac{\partial^{2\,t}\mathtt{U}}{\partial^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}\,\partial^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}}}\Bigg|_{\underline{\underline{\mathbf{0}}}} : \delta^t_\circ\underline{\underline{\boldsymbol{\varepsilon}}} + \cdots \,. \tag{6.58}
$$

Since, in a stable material the value of the elastic strain energy is positive for any strain tensor (the elastic strain energy is a positive-definite function) we conclude that,

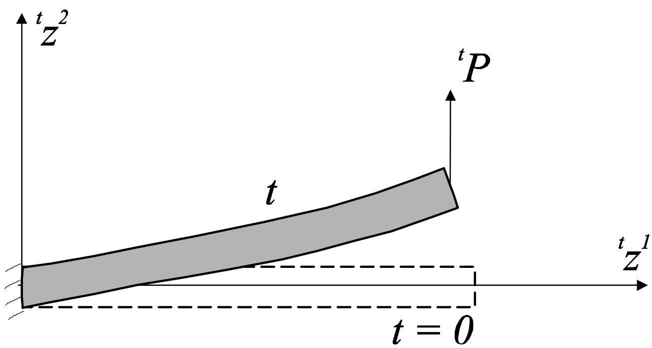
$$
{}^t_o\Pi' - {}^t_o\Pi > 0 \,, \tag{6.59}
$$

and the potential energy is a local minimum at the equilibrium configuration. In the infinitesimal strains case we call it the minimun potential energy principle (Washizu 1982).

Example 6.3. ◀◀◀◀◀
Conservative and nonconservative loading.

(a) Conservative loading

Let us consider a linear elastic, cantilever beam under the conservative end-load shown in the figure,



Conservative load

The elastic energy stored in the beam is,

$$
{}^t\mathtt{U} = \int_0^L \frac{E\,I}{2}\left(\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right)^2 \mathrm{d}^tz_1
$$

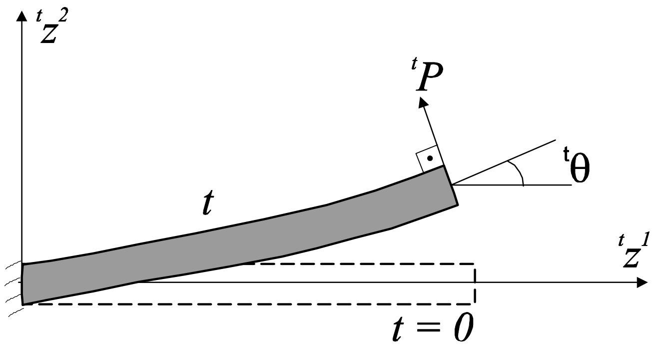
where E is Young's modulus and I is the beam cross section moment of inertia. The Principle of Virtual Work states,

$$\delta^t\mathtt{U} = \ {}^tP\ \delta u_2$$
$$\delta\left({}^t\mathtt{U} - {}^tP\ {}^tu_2\right) = 0$$

where ${}^t\Pi = \ {}^t\mathtt{U}\ -\ {}^tP\ {}^tu_2$ is the potential energy of the system.

(b) Nonconservative loading

We now consider the same linear elastic cantilever beam but under a follower load, as shown in the figure



Body-attached follower load

The principle of virtual work states,

$$\delta^t\mathtt{U} = -\ {}^tP\ \sin\left({}^t\theta\right)\ \delta u_1\ +\ {}^tP\ \cos\left({}^t\theta\right)\ \delta u_2\ .$$

For small displacement derivatives we can approximate

$$\sin\left({}^t\theta\right) \approx {}^t\theta \approx \left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L$$
$$\cos\left({}^t\theta\right) \approx 1$$

hence,

$$\delta^t\mathtt{U} = \ {}^tP\left[-\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L\delta^tu_1 + \delta^tu_2\right]\ .$$

Since

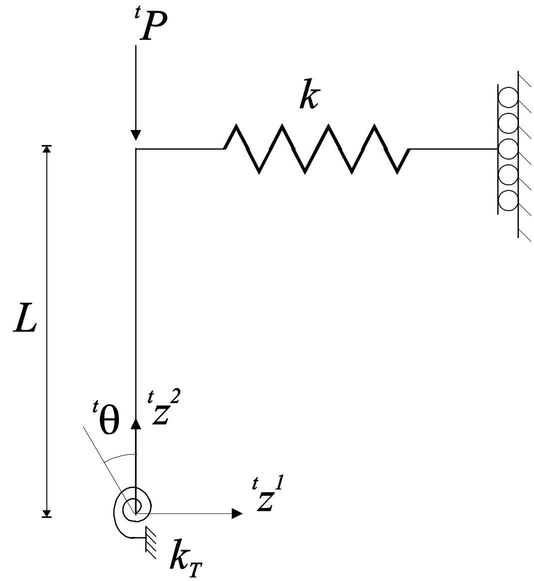
$${}^tP\left[-\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L\delta^tu_1 + \delta^tu_2\right] \neq -\frac{\partial^t\mathtt{G}}{\partial\underline{\mathbf{u}}}\cdot\delta^t\underline{\mathbf{u}}$$

the load is nonconservative and the principle of stationary potential energy cannot be used. ◀◀◀◀◀

Example 6.4. ◀◀◀◀◀

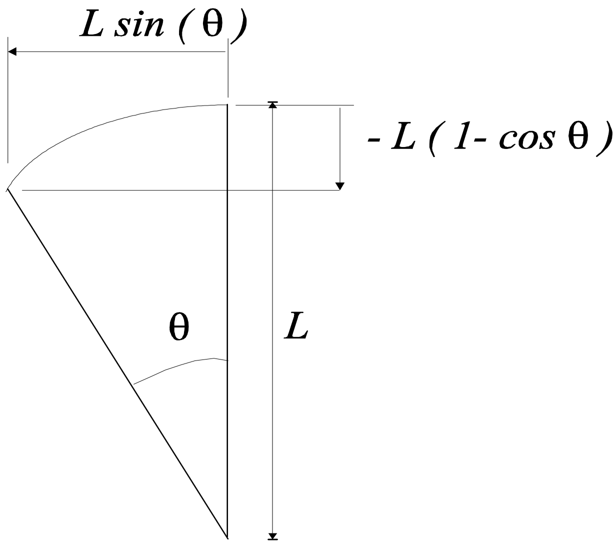
Stability of the equilibrium configuration (buckling) (Hoff 1956).

Let us consider the system shown in the following figure, in equilibrium at time t, in the straight configuration:



${}^tP$: *axial conservative load* ; L : *length of the rigid bar* ; k : *stiffness of the linear spring*; k_T : *stiffness of the torsional spring.*

Assume that the equilibrium configuration is perturbed with a rotation $\theta << 1$. The axial load displacement is obtained from the following scheme:



For $\theta << 1 \Rightarrow \Delta_P \approx -\frac{L\ \theta^2}{2}$ *and* $L\ \sin(\theta) \approx L\ \theta$.

The potential of the external load is,

$$^tG = -\ ^tP\ \Delta_P = \frac{^tP\ L\ \theta^2}{2}\ .$$

The only deformable bodies are the springs; hence

$$^t\mathtt{U} = \frac{1}{2}\ k\ \left(L\ \theta\right)^2\ +\ \frac{1}{2}\ k_T\ \theta^2\ .$$

Therefore the potential energy functional of the system is

$$^t_o\Pi = \frac{1}{2}\ k\ L^2\ \theta^2 + \frac{1}{2}\ k_T\ \theta^2 + \frac{^tP\ L\ \theta^2}{2}$$

and the equilibrium configuration is defined by

$$\delta^t_\circ\Pi = 0$$

which leads to,

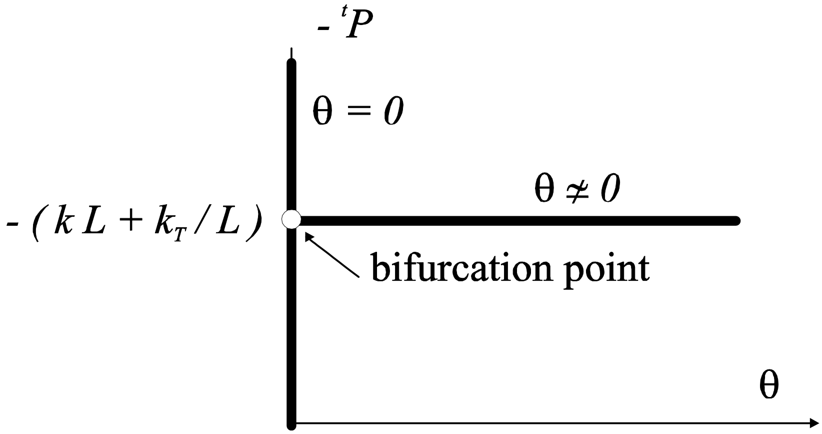
$$\left[k\ L^2\ \theta +\ k_T\ \theta +\ ^tP\ L\ \theta\right]\ \delta\theta = 0\ .$$

Since $\delta\theta$ is arbitrary the bracket has to be zero. Two solutions are possible:

(i) $\theta = 0$; that is to say, the straight (undeformed) configuration.
(ii) $^tP = -\left(k\ L + \frac{k_T}{L}\right)$.

For the second solution θ is undefined. We call this load value the critical value, P_{cr}, because at this load there are two branching solutions ($\theta = 0$ and $\theta \neq 0$).

P_{cr} defines the bifurcation or buckling load.
The equilibrium path is



Since in the above derivation the terms higher than θ^2 were neglected, we cannot assess anything about the branching equilibrium path.

Example 6.5.
Postbuckling behavior.
We repeat the previous example derivation keeping terms higher than θ^2. By doing this, we get

$$\begin{aligned}
\Delta_P &= -\ L\ \ (1-\cos\theta) \approx -\ L\left(\frac{\theta^2}{2}-\frac{\theta^4}{4!}\right)\\
{}^tG &= -\ {}^tP\ \Delta_P \approx\ {}^tP\ L\left(\frac{\theta^2}{2}-\frac{\theta^4}{4!}\right)\\
{}^t\mathtt{U}_k &= \frac{1}{2}\ k\ \ (L\ \sin\theta)^2 \approx \frac{1}{2}\ k\ L^2\ \left(\theta-\frac{\theta^3}{3!}\right)^2\\
{}^t\mathtt{U}_T &= \frac{1}{2}\ k_T\ \theta^2\ .
\end{aligned}$$

Hence

$${}^t_o\Pi = \frac{1}{2}\ k\ L^2\ \left(\theta^2-\frac{\theta^4}{3}+\frac{\theta^6}{36}\right)+\frac{1}{2}\ k_T\ \theta^2+\ {}^tP\ L\ \frac{\theta^2}{2}-\ {}^tP\ L\ \frac{\theta^4}{24}\ .$$

For the equilibrium configuration $\delta^t_\circ\Pi = 0$ and therefore,

$$\left[k\ L^2\ \left(\theta-\frac{2\ \theta^3}{3}+\frac{\theta^5}{12}\right)+k_T\ \theta+\ {}^tP\ L\ \theta-\ {}^tP\ L\ \frac{\theta^3}{6}\right]\ \delta\theta = 0\ .$$

Since $\delta\theta$ is arbitrary, we get, neglecting terms higher than θ^3,

$$\left[k\ L^2\ \left(1-\frac{2\ \theta^2}{3}\right)+k_T\ +\ {}^tP\ L\ \left(1-\frac{\theta^2}{6}\right)\right]\ \theta=0$$

which has again two possible solutions:

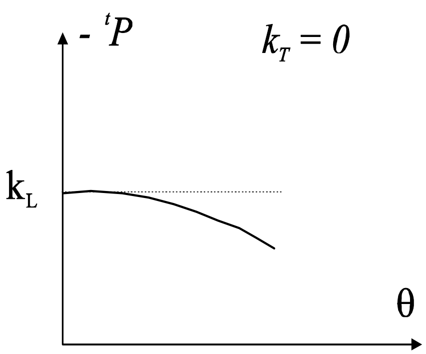
(i) $\theta = 0$ the straight solution

(ii) ${}^tP = -\frac{k\ L\ \left(1-\frac{2\ \theta^2}{3}\right)+\frac{k_T}{L}}{1-\frac{\theta^2}{6}}$

In the second solution, for $\theta = 0$, ${}^tP = P_{cr} = -\left(k\ L + \frac{k_T}{L}\right)$.
The bifurcation point is the same as the one calculated in the previous example; however, now θ is defined.
We see that for $\theta > 0$ Eq. (ii) provides ${}^tP = {}^tP(\theta)$.
If we examine the case with $k_T = 0$, we get

$$-\ {}^tP = \frac{k\ L\ \left(1-\frac{2\ \theta^2}{3}\right)}{1-\frac{\theta^2}{6}}$$

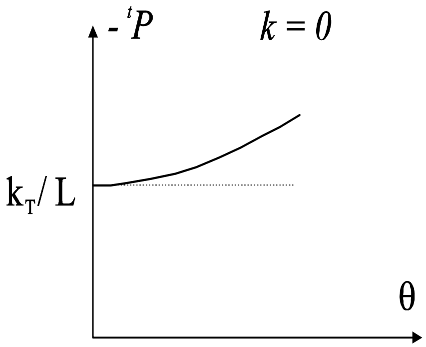
and we can represent



If we examine the case with $k = 0$, we get

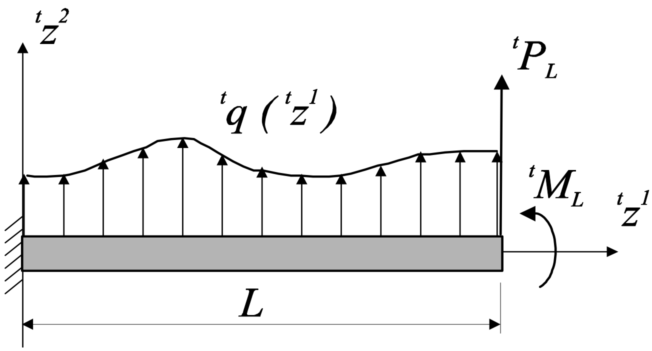
$$-\ {}^tP = \frac{k_T}{L\left(1-\frac{\theta^2}{6}\right)}$$

and we can represent



It is clear that the above cases represent two very different behaviors from a structural point of view.
For the case $k_T = 0$, the buckling is catastrophic because for $\theta > 0$ the load-carrying capacity of the structure keeps dimishing: unstable postbuckling behavior.
For the case $k = 0$, the load-carrying capacity of the structure increases after buckling: stable postbuckling behavior.

Example 6.6.
Natural boundary conditions.
Let us study the following linear elastic cantilever beam under conservative loads:



E : *Young's modulus*; I : *moment of inertia of the beam cross section.*

Assume ${}^tu_2 = {}^t u_2({}^tz_1)$ is the beam transversal displacement and using linear beam theory (Hoff 1956)

$$
\begin{aligned}
{}^t\Pi = \int_0^L \frac{E\,I}{2}\left(\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right)^2 \mathrm{d}^tz_1 - \int_0^L {}^tq\;{}^tu_2\;\mathrm{d}^tz_1 \\
-\;{}^tP_L\;\left({}^tu_2\right)_L - {}^tM_L\;\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L .
\end{aligned}
$$

For the equilibrium configuration $\delta^t\Pi = 0$; hence

$$
\begin{aligned}
\int_0^L E\,I\left(\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right)\delta\left(\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right)\mathrm{d}^tz_1 - \int_0^L {}^tq\;\delta\left({}^tu_2\right)\,\mathrm{d}^tz_1 \\
-\;{}^tP_L\;\delta\left({}^tu_2\right)_L - {}^tM_L\;\delta\left[\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L\right] = 0 .
\end{aligned}
\tag{(A)}
$$

In the first integral we use (Fung 1965) $\delta\left(\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right) = \frac{\mathrm{d}^2}{\mathrm{d}^tz_1{}^2}\left(\delta^tu_2\right)$ and integrating by parts twice, we get

$$
\begin{aligned}
\int_0^L E\,I\left(\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right)\frac{\mathrm{d}^2}{\mathrm{d}^tz_1{}^2}\left(\delta^tu_2\right)\mathrm{d}^tz_1 = \left[E\,I\,\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\,\frac{\mathrm{d}}{\mathrm{d}^tz_1}\left(\delta^tu_2\right)\right]_0^L \\
-\left[E\,I\,\frac{\mathrm{d}^3\,{}^tu_2}{\mathrm{d}^tz_1{}^3}\,\delta^tu_2\right]_0^L + \int_0^L E\,I\,\frac{\mathrm{d}^4\,{}^tu_2}{\mathrm{d}^tz_1{}^4}\,\delta^tu_2\,\mathrm{d}^tz_1 .
\end{aligned}
$$

At ${}^tz_1 = 0$ we have as boundary conditions ${}^tu_2 = \frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1} = 0$, hence, $\left(\delta^tu_2\right)_{{}^tz_1=0} = \left[\delta\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)\right]_{{}^tz_1=0} = 0.$
Replacing in (A), we get

$$
\begin{aligned}
\int_0^L\left[E\,I\,\frac{\mathrm{d}^4\,{}^tu_2}{\mathrm{d}^tz_1{}^4} - {}^tq\right]\delta^tu_2\,\mathrm{d}^tz_1 - \left[\left(E\,I\,\frac{\mathrm{d}^3\,{}^tu_2}{\mathrm{d}^tz_1{}^3}\right)_L + {}^tP_L\right]\delta\left({}^tu_2\right)_L \\
+\left[\left(E\,I\,\frac{\mathrm{d}^2\,{}^tu_2}{\mathrm{d}^tz_1{}^2}\right)_L - {}^tM_L\right]\delta\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L = 0 .
\end{aligned}
$$

Since δ^tu_2 is arbitrary at every point $0 \leqslant {}^tz_1 \leqslant L$ we must fulfill the following differential equation

$$
{}^tq = E\,I\,\frac{\mathrm{d}^4\,{}^tu_2}{\mathrm{d}^tz_1{}^4}
$$

which is the well-known equation of beam theory.
At ${}^tz_1 = L$ we get

Essential (rigid) boundary conditions		Natural boundary condition
Either $\left({}^tu_2\right)_L$ is fixed and $\delta\left({}^tu_2\right)_L = 0$	**or**	${}^tP_L = -\, E\, I\, \left(\frac{\mathrm{d}^3\ {}^tu_2}{\mathrm{d}^tz_1{}^3}\right)_L$
Either $\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L$ is fixed and $\delta\left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L = 0$	**or**	${}^tM_L = \ E\, I\, \left(\frac{\mathrm{d}^2\ {}^tu_2}{\mathrm{d}^tz_1{}^2}\right)_L$

.

Example 6.7.
The Rayleigh-Ritz method.
In the previous example, from the principle of stationary potential energy we derived the differential equation that governs the deformation of a cantilever beam.
Usually, we need to work in the opposite direction: we know the differential equations that govern the deformation of a continuum but we cannot integrate them and we resort to the principle of stationary potential energy to derive an approximate solution. One method for deriving approximate solutions is the Rayleigh-Ritz method (Hoff 1956).
Let us consider again the linear case analyzed in the previous example and let us assume that we want to derive an approximate solution for the case ${}^tq \ = \ {}^tP_L = 0$. For this case

$$ {}^t\Pi = \int_0^L \frac{E\ I}{2}\left(\frac{\mathrm{d}^2\ {}^tu_2}{\mathrm{d}^tz_1{}^2}\right)^2 \mathrm{d}^tz_1 - \ {}^tM_L\ \left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_L \ . $$

To derive an approximate solution we consider trial functions that fulfill the geometrical or essential boundary conditions,

$$ {}^tu_2(0) = \left(\frac{\mathrm{d}^tu_2}{\mathrm{d}^tz_1}\right)_0 = 0\ . $$

For example

$$ {}^tu_2^{\ominus}({}^tz_1) = a\left(1-\cos\frac{\pi\ {}^tz_1}{2L}\right) $$

where the parameter a will be determined by imposing the minimization condition on

$$ {}^t\Pi^{\ominus} = \ {}^t\Pi^{\ominus}(a)\ . $$

Using the adopted trial function, we get

$$ {}^t\Pi^{\ominus} = \frac{E\ I\ \pi^4\ a^2}{64\ L^3} - \frac{{}^tM_L\ a\ \pi}{2\ L}\ . $$

The minimum value that can attain the above functional is, within the considered set of trial functions, our best approximation to the equilibrium configuration. Imposing

$$\frac{\partial^t \Pi^{\ominus}}{\partial a} = 0 \ ,$$

we get

$$a = \frac{16\ {}^t M_L\ L^2}{E\ I\ \pi^3} \ .$$

Therefore our approximate solution is

$$\begin{aligned} {}^t u_2^{\ominus}({}^t z_1) &= \frac{16\ {}^t M_L\ L^2}{E\ I\ \pi^3}\left(1 - \cos\frac{\pi^t z_1}{2L}\right) \\ {}^t \Pi^{\ominus} &= -\frac{4\ {}^t M_L^2\ L}{E\ I\ \pi^2} \ . \end{aligned}$$

For the case we are analyzing the exact solution is

$$\begin{aligned} {}^t u_2{}^{exact}\left({}^t z_1\right) &= \frac{{}^t M_L\ \left({}^t z_1\right)^2}{2\ E\ I} \\ {}^t \Pi^{exact} &= -\frac{{}^t M_L^2\ L}{2\ E\ I} \ . \end{aligned}$$

It is obvious that ${}^t \Pi^{\ominus} \ > \ {}^t \Pi^{exact}$.

If we want to improve our approximate solution we enrich the trial function set using, for example

$${}^t u_2{}^{\oplus}\left({}^t z_1\right) = a\left(1 - \cos\frac{\pi^t z_1}{2L}\right) + b\left(1 - \cos\frac{\pi^t z_1}{L}\right) \ .$$

It is important to note that the above defined trial function:

∘ Fulfills the essential (rigid) boundary conditions.

∘ Contains the previous one, ${}^t u_2{}^{\ominus}({}^t z_1)$, as a particular case ($b = 0$).

Since we will determine the values of both constants by imposing on ${}^t \Pi^{\oplus}$ the necessary conditions for attaining a minimum, it is obvious that

$${}^t \Pi^{\oplus} \ \leqslant \ {}^t \Pi^{\ominus} \ .$$

That is to say, we will either find the same solution as before ($b = 0$ and ${}^t \Pi^{\oplus} = {}^t \Pi^{\ominus}$) or a better one ($b \neq 0$ and ${}^t \Pi^{\oplus} < {}^t \Pi^{\ominus}$). We cannot deteriorate the solution by adding more terms in the trial function.

Using ${}^t u_2{}^{\oplus}({}^t z_1)$, we get

$${}^t \Pi^{\oplus} = \frac{E\ I\ \pi^3}{2\ L^3}\left(\frac{a\ b}{3} + \frac{\pi}{32}a^2 + \frac{\pi}{2}b^2\right) - \frac{{}^t M_L\ a\ \pi}{2\ L} \ .$$

Imposing $\frac{\partial^t \Pi^{\oplus}}{\partial a} = \frac{\partial^t \Pi^{\oplus}}{\partial b} = 0$, we get

$$a = 0.6294 \; \frac{{}^tM_L \; L^2}{E \; I}$$
$$b = -0.0668 \; \frac{{}^tM_L \; L^2}{E \; I}$$
$${}^t\Pi^{\oplus} = -0.4940 \; \frac{{}^tM_L^2 \; L}{E \; I} \; .$$

It is clear from the derived values that, as expected: ${}^t\Pi^{exact} < {}^t\Pi^{\oplus} < {}^t\Pi^{\ominus}$, and therefore ${}^tu_2{}^{\oplus}\,({}^tz_1)$ is a "better" approximation than ${}^tu_2{}^{\ominus}\,({}^tz_1)$. ◀◀◀◀◀

In the previous example we have introduced three relevant topics that we want to highlight:

1. When obtaining approximate solutions using the Rayleigh-Ritz method, based on the minimum potential energy principle (infinitesimal strains), we can only rank the merit of different solution using their potential energy value; that is to say, if ${}^t\Pi^A < {}^t\Pi^B$ then the A-solution is a better approximation than the B-solution.
2. The trial functions have to exactly satisfy the rigid boundary conditions (admissible functions) but not the natural boundary conditions.
3. Approximate solutions do not need to fulfill exactly either the equilibrium equation inside the dominium or the natural boundary conditions (equilibrium equations on the boundary).

6.5 Kinematic constraints

In the example shown in Fig. 6.4, where ${}^tP$ is a conservative load, the potential energy is,

$${}^t\Pi = \frac{1}{2}\,k\,{}^tu^2 - {}^tP\,{}^tv \; . \qquad (6.60)$$

For the inextensible string there is a kinematic constraint given by,

$$2\,{}^tv - {}^tu = 0 \; . \qquad (6.61)$$

Hence, we have to minimize the functional of the potential energy given in Eq.(6.60) under the constraint expressed in Eq. (6.61). Using the Lagrange multipliers technique (Fung 1965, Fung & Tong 2001), we define a new functional (${}^t\Pi^*$) and we perform on it an unconstrained minimization:

$${}^t\Pi^* = \frac{1}{2}\,k\,{}^tu^2 - {}^tP\,{}^tv + {}^t\lambda\left(2\,{}^tv - {}^tu\right) \qquad (6.62)$$

where ${}^t\lambda$ is the Lagrange multiplier.

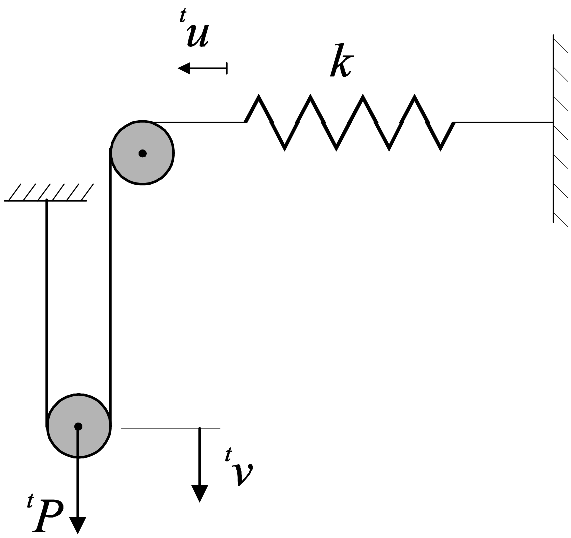


Fig. 6.4. Kinematics constraints

We need to determine the set $({}^t\lambda,\ {}^t u,\ {}^t v)$ that satisfies $\delta^t \Pi^* = 0$. The necessary conditions are,

$$\frac{\partial\, {}^t\Pi^*}{\partial\, {}^t\lambda} = 0 \tag{6.63a}$$

$$\frac{\partial\, {}^t\Pi^*}{\partial\, {}^t u} = 0 \tag{6.63b}$$

$$\frac{\partial\, {}^t\Pi^*}{\partial\, {}^t v} = 0\ . \tag{6.63c}$$

From the above we get,

$${}^t u = \frac{{}^t P}{2\ k} \tag{6.64a}$$

$${}^t v = \frac{{}^t P}{4\ k} \tag{6.64b}$$

$${}^t\lambda = \ k\ {}^t u = \frac{{}^t P}{2}\ . \tag{6.64c}$$

Example 6.8. ——————————————————————— ◀◀◀◀◀

Physical interpretation of the Lagrange multiplier in the above example (Crandall 1956).

Let us assume that the string, instead of being inextensible has a stiffness k_s; hence, with ${}^t\lambda$ the tensile load on the spring we have,

$${}^t\lambda = k_s\ \left(2\ {}^t v -\ {}^t u\right)\ .$$

For this system with no constraints we can define the potential energy

$$^{t}\Pi_{k_s} = \frac{1}{2}\ k\ {}^{t}u^2\ +\ \frac{1}{2}\ k_s\ \left(2\ {}^{t}v -\ {}^{t}u\right)^2\ -\ {}^{t}P\ {}^{t}v$$

and $\delta^t \Pi_{k_s} = 0$ leads to,

$$\begin{aligned} k\ {}^{t}u\ -\ k_s\ \left(2\ {}^{t}v -\ {}^{t}u\right)\ &=\ 0 \\ 2\ k_s\ \left(2\ {}^{t}v -\ {}^{t}u\right)\ -\ {}^{t}P\ &= 0\ . \end{aligned}$$

Hence,

$$^{t}u = \frac{{}^{t}P}{2\ k} \qquad ; \qquad {}^{t}\lambda = \frac{{}^{t}P}{2}\ .$$

Also,

$$2\ {}^{t}v -\ {}^{t}u = \frac{{}^{t}\lambda}{k_s} \quad \underset{k_s \to \infty}{\longrightarrow} \quad 0\ .$$

Therefore, ${}^{t}\lambda$ is independent of the displacement (conservative load) and we can use Eq. (6.62) for writing the potential energy, and identify the Lagrangian multiplier with the string tensile load. ◀◀◀◀◀

In the above example, it is important to realize that the Lagrangian multiplier is the energy conjugate of the physical magnitude that represents the constraint equation: the string elongation.

6.6 Veubeke-Hu-Washizu variational principles

In the previous section we analyzed a simple mechanical system and we developed the imposition, using the Lagrange multipliers technique, of a kinematic constraint on the stationarization of the potential energy functional.

In the present section we are going to generalize the above technique to impose different constraints on the potential energy functional (Fraeijs de Veubeke 1965).

6.6.1 Kinematic constraints via the V-H-W principles

Let us assume that we relax the compatibility conditions (see Sect. 2.15) when formulating the potential energy functional; that is to say we consider in Eq. (6.51) a strain tensor ${}^{t}_{o}\underline{\underline{\varepsilon}}$ that can not necessarily be derived from the displacements field.

We now define the functional:

$$\begin{aligned} {}^{t}_{o}\Pi^{*} = \int_{{}^{\circ}V} {}^{o}\rho\ \left[\ {}^{t}\mathrm{U}\left({}^{t}_{o}\underline{\underline{\varepsilon}}\right) +\ {}^{t}G({}^{t}\underline{\mathbf{u}})\right]\ {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}} {}^{t}g\left({}^{t}\underline{\mathbf{u}}\right)\ {}^{\circ}\mathrm{d}S \\ + \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\lambda}}} : \left({}^{t}_{o}\underline{\underline{\boldsymbol{\varepsilon}}}\left({}^{t}\underline{\mathbf{u}}\right) -{}^{t}_{o}\underline{\underline{\varepsilon}}\right)\ {}^{\circ}\mathrm{d}V \end{aligned} \qquad (6.65)$$

and we search for the equilibrium configuration imposing

$$\delta^t_\circ \Pi^* = 0 \tag{6.66}$$

we have as independent variables: ${}^t\underline{\mathbf{u}}$, ${}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}$ and ${}^t\underline{\underline{\lambda}}$; with ${}^t_o\overline{\underline{\underline{\mathbf{S}}}} = {}^o\rho \frac{\partial^t \mathbb{U}}{\partial^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}}$ we get

$$\begin{aligned} &\int_{\circ V} {}^t_o\overline{\underline{\underline{\mathbf{S}}}} : \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}\, {}^\circ\mathrm{d}V - \int_{\circ V} {}^o\rho\, {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}V - \int_{\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}S \\ &+ \int_{\circ V} \delta^t\underline{\underline{\lambda}} : ({}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} - {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}})\, {}^\circ\mathrm{d}V + \int_{\circ V} {}^t\underline{\underline{\boldsymbol{\lambda}}} : (\delta\, {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} - \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}})\, {}^\circ\mathrm{d}V = 0\,. \end{aligned} \tag{6.67}$$

Considering that the variations are arbitrary,

$$\int_{\circ V} {}^t\underline{\underline{\boldsymbol{\lambda}}} : \delta\, {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}}\, {}^\circ\mathrm{d}V = \int_{\circ V} {}^o\rho\, {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}V + \int_{\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}S \tag{6.68}$$

$$\int_{\circ V} \left({}^t_o\overline{\underline{\underline{\mathbf{S}}}} - {}^t\underline{\underline{\boldsymbol{\lambda}}} \right) : \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}\, {}^\circ\mathrm{d}V = 0 \tag{6.69}$$

$$\int_{\circ V} \delta^t\underline{\underline{\boldsymbol{\lambda}}} : ({}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} - {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}})\, {}^\circ\mathrm{d}V = 0 \quad . \tag{6.70}$$

The last equations impose the conditions

$${}^t_o\overline{\underline{\underline{\mathbf{S}}}} = {}^t\underline{\underline{\boldsymbol{\lambda}}} \tag{6.71}$$

$${}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}} = {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} \tag{6.72}$$

which are always fulfilled for the continuum problem but not necessarily in finite element approximations.

Adding Eqs. (6.68) and (6.69), we get

$$\begin{aligned} &\int_{\circ V} {}^t_o\overline{\underline{\underline{\mathbf{S}}}} : \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}\, {}^\circ\mathrm{d}V + \int_{\circ V} {}^t\underline{\underline{\boldsymbol{\lambda}}} : (\delta\, {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} - \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}})\, {}^\circ\mathrm{d}V \\ &= \int_{\circ V} {}^o\rho\, {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}V + \int_{\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}S \quad . \end{aligned} \tag{6.73}$$

From the above, we can state the Principle of Virtual Work as,

$$\int_{\circ V} {}^t_o\overline{\underline{\underline{\mathbf{S}}}} : \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}\, {}^\circ\mathrm{d}V = \int_{\circ V} {}^o\rho\, {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}V + \int_{\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\, {}^\circ\mathrm{d}S \tag{6.74}$$

as long as we fulfill the condition of variational consistency (Simo & Hughes 1986),

$$\int_{\circ V} {}^t\underline{\underline{\boldsymbol{\lambda}}} : (\delta\, {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}} - \delta\, {}^t_o\overline{\underline{\underline{\boldsymbol{\varepsilon}}}})\, {}^\circ\mathrm{d}V = 0\,, \tag{6.75}$$

which is obviously fulfilled for the continuum problem.

6.6.2 Constitutive constraints via the V-H-W principles

Let us now assume that we consider in Eq. (6.51) a stress tensor ${}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}}$ that is not necessarily derived from the kinematically consistent strain field ${}^{t}_{o}\underline{\underline{\boldsymbol{\varepsilon}}}$.

We can write

$$ {}^{t}_{o}\underline{\underline{\mathbf{S}}} = {}^{o}\rho \frac{\partial^{t}\mathrm{U}}{\partial^{t}_{o}\underline{\underline{\boldsymbol{\varepsilon}}}} \tag{6.76a} $$

$$ {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}} = {}^{o}\rho \frac{\partial^{t}\mathrm{U}}{\partial^{t}_{o}\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}} \ . \tag{6.76b} $$

Therefore

$$ \begin{aligned} {}^{t}_{o}\Pi^{*} = \int_{{}^{\circ}V} {}^{o}\rho \ \left[{}^{t}\mathrm{U}\left({}^{t}_{o}\overline{\underline{\underline{\boldsymbol{\varepsilon}}}}\right) + {}^{t}G({}^{t}\underline{\mathbf{u}})\right] \ {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}} {}^{t}g\left({}^{t}\underline{\mathbf{u}}\right) \ {}^{\circ}\mathrm{d}S \\ + \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\lambda}}} : \left({}^{t}_{o}\underline{\underline{\mathbf{S}}} - {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}}\right) {}^{\circ}\mathrm{d}V \quad . \end{aligned} \tag{6.77} $$

We search for the equilibrium configuration imposing

$$ \delta^{t}_{\circ}\Pi^{*} = 0 \ , \tag{6.78} $$

and considering the independent variables ${}^{t}\underline{\mathbf{u}}, {}^{t}\underline{\underline{\boldsymbol{\lambda}}}$ and $\left({}^{t}_{o}\overline{\underline{\underline{\boldsymbol{\varepsilon}}}} \ , \ {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}}\right)$:

$$ \int_{{}^{\circ}V} {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}} : \delta \, {}^{t}_{o}\overline{\underline{\underline{\boldsymbol{\varepsilon}}}} \ {}^{\circ}\mathrm{d}V - \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\lambda}}} : \delta^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}} \ {}^{\circ}\mathrm{d}V = 0 \tag{6.79} $$

$$ \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\lambda}}} : \delta^{t}_{o}\underline{\underline{\mathbf{S}}} \ {}^{\circ}\mathrm{d}V - \int_{{}^{\circ}V} {}^{o}\rho \ {}^{t}\underline{\mathbf{b}} \ \cdot \delta^{t}\underline{\mathbf{u}} \ {}^{\circ}\mathrm{d}V \ - \int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{u}} \ {}^{\circ}\mathrm{d}S = 0 \tag{6.80} $$

$$ \int_{{}^{\circ}V} \delta^{t}\underline{\underline{\boldsymbol{\lambda}}} : \left({}^{t}_{o}\underline{\underline{\mathbf{S}}} - {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}}\right) {}^{\circ}\mathrm{d}V = 0 \quad . \tag{6.81} $$

The last equation imposes the condition

$$ {}^{t}_{o}\underline{\underline{\mathbf{S}}} = {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}} \tag{6.82} $$

which is always fulfilled for the continuum problem but not necessarilly in the finite element approximations.

Adding Eqs. (6.79) and (6.80), we get

$$ \begin{aligned} \int_{{}^{\circ}V} {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}} : \delta \, {}^{t}_{o}\overline{\underline{\underline{\boldsymbol{\varepsilon}}}} \ {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}V} {}^{t}\underline{\underline{\boldsymbol{\lambda}}} : \left(\delta \, {}^{t}_{o}\underline{\underline{\mathbf{S}}} - \delta \, {}^{t}_{o}\overline{\underline{\underline{\mathbf{S}}}}\right) {}^{\circ}\mathrm{d}V \\ = \int_{{}^{\circ}V} {}^{o}\rho \ {}^{t}\underline{\mathbf{b}} \ \cdot \delta^{t}\underline{\mathbf{u}} \ {}^{\circ}\mathrm{d}V \ + \int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta^{t}\underline{\mathbf{u}} \ {}^{\circ}\mathrm{d}S \ . \end{aligned} \tag{6.83} $$

From the above, we can state the Principle of Virtual Work as,

$$\int_{^\circ V} {}^t_o\underline{\underline{\overline{\mathbf{S}}}} : \delta\, {}^t_o\underline{\underline{\overline{\boldsymbol{\varepsilon}}}}\ {}^\circ\mathrm{d}V = \int_{^\circ V} {}^o\rho\ {}^t\underline{\mathbf{b}} \cdot \delta{}^t\underline{\mathbf{u}}\ {}^\circ\mathrm{d}V \ + \int_{^\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta{}^t\underline{\mathbf{u}}\ {}^\circ\mathrm{d}S \qquad (6.84)$$

as long as we fulfill the condition,

$$\int_{^\circ V} {}^t\underline{\underline{\boldsymbol{\lambda}}} : (\delta\ {}^t_o\underline{\underline{\mathbf{S}}} - \delta\ {}^t_o\underline{\underline{\overline{\mathbf{S}}}})\ {}^\circ\mathrm{d}V = 0\ , \qquad (6.85)$$

which is obviously fulfilled for the continuum problem.

Other constraints can also be considered and they constitute the basis of different finite element applications.

When using variational principles of the Veubeke-Hu-Washizu type for developing finite element formulations, the interpolation functions selected for the different interpolated fields should fulfill orthogonality conditions of the form of Eqs. (6.75) or (6.85) (Simo & Hughes 1986).

Different forms of the Veubeke-Hu-Washizu variational principles have been used to develop mixed and hybrid finite element formulation some of them can be read from the classical paper (Pian & Tong 1969) and also from (Dvorkin & Bathe 1984, Bathe & Dvorkin 1985\1986, Dvorkin & Vassolo 1989, Fung & Tong 2001), etc.

A

Introduction to tensor analysis

In this Appendix, assuming that the reader is acquainted with vector analysis, we present a short introduction to tensor analysis. However, since tensor analysis is a fundamental tool for understanding continuum mechanics, we strongly recommend a deeper study of this subject.

Some of the books that can be used for that purpose are: (Synge & Schild 1949, McConnell 1957, Santaló 1961, Aris 1962, Sokolnikoff 1964, Fung 1965, Green & Zerna 1968, Flügge 1972, Chapelle & Bathe 2003).

A.1 Coordinates transformation

Let us assume that in a three-dimensional space ($\Re^3$) we can define a system of Cartesian coordinates: we call this space the *Euclidean space.*

In this Appendix we will restrict our presentation to the case of the Euclidean space.

In the $\Re^3$ space we define a system of Cartesian coordinates $\{ \ z^\alpha \ , \ \alpha = 1,2,3 \ \}$, and an arbitrary system of *curvilinear coordinates* $\{ \ \theta^i \ , \ i = 1,2,3 \ \}$. The following relations hold:

$$\theta^i \ = \ \theta^i(z^\alpha \ , \ \alpha = 1,2,3) \qquad , \qquad i = 1,2,3 \ . \tag{A.1}$$

The above functions are single-valued, continuous and with continuous first derivatives.

We call J the *Jacobian* of the coordinates transformation defined by Eq. (A.1). Hence

$$J \ = \ \left[\frac{\partial \theta^i}{\partial z^\alpha} \right] \ . \tag{A.2}$$

An *admissible transformation* is one in which $\det J \neq 0$, that is to say, a transformation where a region of nonzero volume in one system does not collapse into a point in the other system and vice versa.

A *proper transformation* is an admissible transformation in which $\det J > 0$.

A.1.1 Contravariant transformation rule

From Eq. (A.1) we obtain

$$\mathrm{d}\theta^i = \frac{\partial \theta^i}{\partial z^\alpha} \mathrm{d}z^\alpha \ . \tag{A.3}$$

When the coordinates system is changed, the mathematical entities a^i at a certain point of $\Re^3$ that transform following the same rule as does the coordinate differentials (Eq. (A.3)) are said to transform according to a *contravariant transformation rule*. We indicate these mathematical entities using upper indices.

Now we consider two systems of curvilinear coordinates $\{\theta^i\}$ and $\{\hat{\theta}^i\}$, related by the following equations:

$$\hat{\theta}^i = \hat{\theta}^i(\theta^j \ , \ j = 1,2,3) \qquad , \qquad i = 1,2,3 \tag{A.4a}$$

and

$$\theta^k = \theta^k(\hat{\theta}^l \ , \ l = 1,2,3) \qquad , \qquad k = 1,2,3 \ . \tag{A.4b}$$

We can write the coordinate differentials as:

$$\mathrm{d}\theta^i = \frac{\partial \theta^i}{\partial \hat{\theta}^j} \mathrm{d}\hat{\theta}^j \tag{A.4c}$$

$$\mathrm{d}\hat{\theta}^i = \frac{\partial \hat{\theta}^i}{\partial \theta^j} \mathrm{d}\theta^j \ . \tag{A.4d}$$

In the same way, a contravariant mathematical entity can be defined in either of the two systems

$$a^i = a^i(\theta^j \ , \ j = 1,2,3) \qquad , \qquad i = 1,2,3 \tag{A.4e}$$

$$\hat{a}^i = \hat{a}^i(\hat{\theta}^j \ , \ j = 1,2,3) \qquad , \qquad i = 1,2,3 \tag{A.4f}$$

and we transform it from one curvilinear system to the other following a transformation rule similar to the transformation rule followed by the coordinate differentials:

$$a^i = \frac{\partial \theta^i}{\partial \hat{\theta}^j} \hat{a}^j \qquad , \qquad i = 1,2,3 \tag{A.4g}$$

$$\hat{a}^i = \frac{\partial \hat{\theta}^i}{\partial \theta^j} a^j \qquad , \qquad i = 1,2,3 \ . \tag{A.4h}$$

> Although the contravariant transformation rule applies to $\mathrm{d}\theta^i$ and not to θ^i, using a notation abuse, we follow the convention of using upper indices for the coordinates.

A.1.2 Covariant transformation rule

Given an arbitrary continuous and differentiable function $f(\theta^1, \theta^2, \theta^3)$ and using the chain rule, we write

$$\frac{\partial f}{\partial \hat{\theta}^i} = \frac{\partial f}{\partial \theta^j} \frac{\partial \theta^j}{\partial \hat{\theta}^i} \quad , \quad i = 1, 2, 3 \ . \tag{A.5a}$$

We define

$$a_j = \frac{\partial f}{\partial \theta^j} \quad , \quad j = 1, 2, 3 \ . \tag{A.5b}$$

In the $\{\hat{\theta}^i\}$ coordinate system we define

$$\hat{a}_j = \frac{\partial f}{\partial \hat{\theta}^j} \quad , \quad j = 1, 2, 3 \tag{A.5c}$$

$$\hat{a}_i = a_j \frac{\partial \theta^j}{\partial \hat{\theta}^i} \quad , \quad i = 1, 2, 3 \tag{A.5d}$$

$$a_i = \hat{a}_j \frac{\partial \hat{\theta}^j}{\partial \theta^i} \quad , \quad i = 1, 2, 3 \ . \tag{A.5e}$$

When the coordinates system is changed, the mathematical entities a_i at a certain point of $\Re^3$that transform following the same rule as does the derivatives of a scalar function (Eqs. (A.5d) and (A.5e)) are said to transform according to a *covariant transformation rule.* We indicate those mathematical entities using lower indices.

A.2 Vectors

There are some physical properties like mass, temperature, concentration of a given substance, etc., whose values do not change when the coordinate system used to describe the problem is changed. These variables are referred to as *scalars.*

On the other hand, there are other physical variables like velocity, acceleration, force, etc. that do not change their intensity and direction when the coordinate system used to describe the problem is changed. They are called *vectors.*

In what follows, we will make use of the above intuitive definition of scalars and vectors. However, in Sect. A.4 we will see that they represent two particular kinds of tensors (order 0 and 1, respectively).

A.2.1 Base vectors

A set of n *linearly independent vectors* is a *basis* of the space $\Re^n$ and any other vector in $\Re^n$ can be constructed as a linear combination of those base vectors.

Let us consider the three linearly independent vectors $\underline{\mathbf{g}}_i \ \ (i = 1, 2, 3)$ in $\Re^3$. Any vector $\underline{\mathbf{v}}$ in the same space can be written as:

$$\underline{\mathbf{v}} \ = \ v^i \, \underline{\mathbf{g}}_i \ . \tag{A.6}$$

The mathematical entities $v^i \ (i = 1, 2, 3)$ are the *components* of $\underline{\mathbf{v}}$ in the basis $\underline{\mathbf{g}}_i \ \ (i = 1, 2, 3)$.

Example A.1. ◀◀◀◀◀

In a Cartesian system $\{z^\alpha \ , \ \alpha = 1, 2, 3\}$ the base vectors are

$$\begin{aligned} \underline{\mathbf{e}}_1 &= (1, 0, 0) \ , \\ \underline{\mathbf{e}}_2 &= (0, 1, 0) \ , \\ \underline{\mathbf{e}}_3 &= (0, 0, 1) \ , \end{aligned}$$

where we have indicated the projection of the base vectors on the Cartesian axes.

The position vector $\underline{\mathbf{r}}$ of a point P in $\Re^3$ is

$$\underline{\mathbf{r}} \ = \ z^\alpha \, \underline{\mathbf{e}}_\alpha \ .$$

Hence,

$$\mathrm{d}\underline{\mathbf{r}} = \mathrm{d}z^\alpha \underline{\mathbf{e}}_\alpha \ ,$$

but also,

$$\mathrm{d}\underline{\mathbf{r}} = \frac{\partial \underline{\mathbf{r}}}{\partial z^\alpha} \mathrm{d}z^\alpha \ .$$

Therefore, we get

$$\underline{\mathbf{e}}_\alpha \ = \ \frac{\partial \underline{\mathbf{r}}}{\partial z^\alpha} \qquad , \ \ \alpha = 1, 2, 3 \ .$$

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A.2.2 Covariant base vectors

In the arbitrary curvilinear system $\{\theta^i \ , \ i = 1, 2, 3\}$ we can write, at any point P of the space,

$$\mathrm{d}\underline{\mathbf{r}} = \frac{\partial \underline{\mathbf{r}}}{\partial \theta^i} \mathrm{d}\theta^i \ . \tag{A.7a}$$

Since

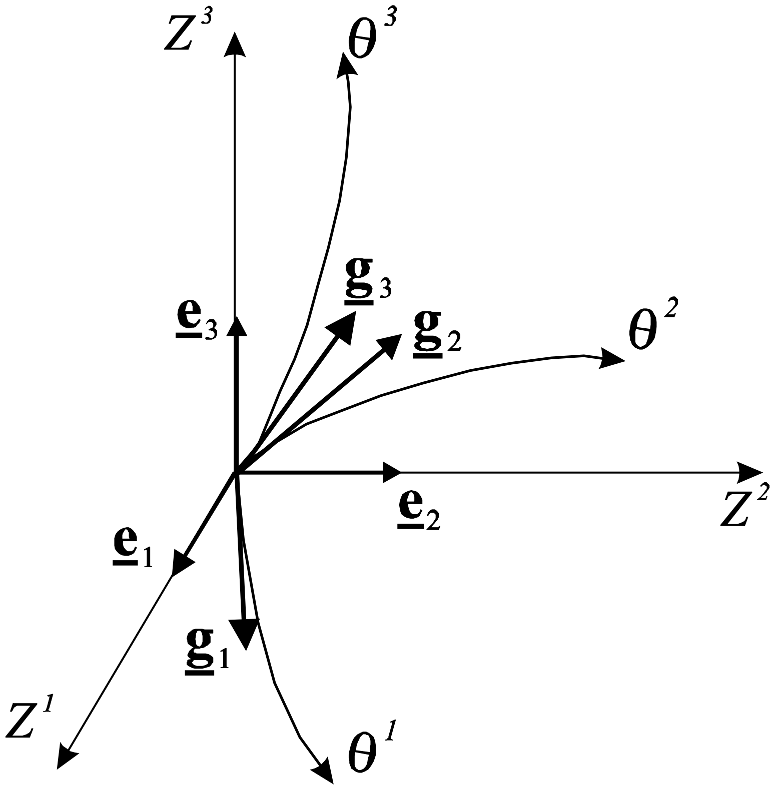

Fig. A.1. Covariant base vectors at a point P

$$\mathrm{d}\underline{\mathbf{r}} = \mathrm{d}\theta^i \underline{\mathbf{g}}_i \tag{A.7b}$$

we obtain

$$\underline{\mathbf{g}}_i \;=\; \frac{\partial \underline{\mathbf{r}}}{\partial \theta^i} \qquad , \quad \alpha = 1, 2, 3 \ . \tag{A.7c}$$

The vectors $\underline{\mathbf{g}}_i$, defined with the above equation, are the *covariant base vectors* of the curvilinear coordinate system $\{\ \theta^i\ \}$ at the point P.

From its definition, the vector $\underline{\mathbf{g}}_1$ is tangent to the line, $\theta_2 \;=\; \theta_2\ (P)$ and $\theta_3 \;=\; \theta_3\ (P)$.

Similar conclusions can be reached for the covariant base vectors $\underline{\mathbf{g}}_2$ and $\underline{\mathbf{g}}_3$.

In a Cartesian system, we can write Eq. (A.7c) as:

$$\underline{\mathbf{g}}_i \;=\; \frac{\partial z^\alpha}{\partial \theta^i}\ \underline{\mathbf{e}}_\alpha \qquad , \quad i = 1, 2, 3 \ . \tag{A.8}$$

In a second curvilinear system $\{\ \hat{\theta}^i\ ,\ i = 1, 2, 3\ \}$,

$$\mathrm{d}\underline{\mathbf{r}} = \mathrm{d}\hat{\theta}^i \underline{\hat{\mathbf{g}}}_i = \mathrm{d}\theta^i \frac{\partial \hat{\theta}^j}{\partial \theta^i} \underline{\hat{\mathbf{g}}}_j \ . \tag{A.9a}$$

Hence, we have

$$\underline{\mathbf{g}}_i \ = \ \frac{\partial \hat{\theta}^j}{\partial \theta^i} \ \underline{\hat{\mathbf{g}}}_j \qquad , \quad i = 1,2,3 \ . \tag{A.9b}$$

Due to the similarity between Eqs. (A.9b) and Eq. (A.5e) the base vectors $\underline{\mathbf{g}}_i$ are called covariant base vectors.

A.2.3 Contravariant base vectors

In an arbitrary curvilinear coordinate system $\{ \ \theta^i \ , \ i = 1,2,3 \ \}$ we define the *contravariant base vectors* (dual basis) ($\underline{\mathbf{g}}^i \ , \ i = 1,2,3$) with the equation

$$\underline{\mathbf{g}}^i \ \cdot \ \underline{\mathbf{g}}_j \ = \ \delta^i_j \ , \tag{A.10}$$

where the dot indicates a scalar product ("dot product") between two vectors and δ^i_j is the Kronecker delta ($\delta^i_j \ = \ 1$ for $i = j$ and $\delta^i_j \ = \ 0$ for $i \neq j$).

Defining in $\Re^3$ two curvilinear systems $\{\theta^i\}$ and $\{\hat{\theta}^i\}$ and using Eq. (A.9b), we obtain

$$\underline{\hat{\mathbf{g}}}^i \ \cdot \ \underline{\hat{\mathbf{g}}}_j \ = \ \underline{\hat{\mathbf{g}}}^i \ \cdot \ \frac{\partial \theta^m}{\partial \hat{\theta}^j} \ \underline{\mathbf{g}}_m \ = \ \delta^i_j \ . \tag{A.11a}$$

Hence, using Eq. (A.10), we obtain

$$\underline{\mathbf{g}}^i \ \cdot \ \underline{\mathbf{g}}_j \ = \ \underline{\hat{\mathbf{g}}}^i \ \cdot \ \frac{\partial \theta^m}{\partial \hat{\theta}^j} \ \underline{\mathbf{g}}_m \ . \tag{A.11b}$$

If we define

$$\underline{\hat{\mathbf{g}}}^i \ = \ \frac{\partial \hat{\theta}^i}{\partial \theta^l} \ \underline{\mathbf{g}}^l \qquad , \quad i = 1,2,3 \tag{A.11c}$$

from Eq. (A.11b), we obtain

$$\underline{\mathbf{g}}^i \ \cdot \ \underline{\mathbf{g}}_j \ = \ \frac{\partial \hat{\theta}^i}{\partial \theta^l} \ \frac{\partial \theta^m}{\partial \hat{\theta}^j} \ \underline{\mathbf{g}}^l \ \cdot \ \underline{\mathbf{g}}_m \tag{A.11d}$$

and

$$\underline{\mathbf{g}}^i \ \cdot \ \underline{\mathbf{g}}_j \ = \ \frac{\partial \hat{\theta}^i}{\partial \theta^l} \ \frac{\partial \theta^m}{\partial \hat{\theta}^j} \ \delta^l_m \ = \ \frac{\partial \hat{\theta}^i}{\partial \hat{\theta}^j} \ = \ \delta^i_j \ , \tag{A.11e}$$

where we can see that the relation (A.11a) is satisfied.

Therefore, Eq.(A.11c) can be considered the transformation rule for the contravariant base vectors. Due to the similarity between Eqs. (A.11c) and (A.4h) the base vectors $\underline{\mathbf{g}}^i$ are called contravariant base vectors.

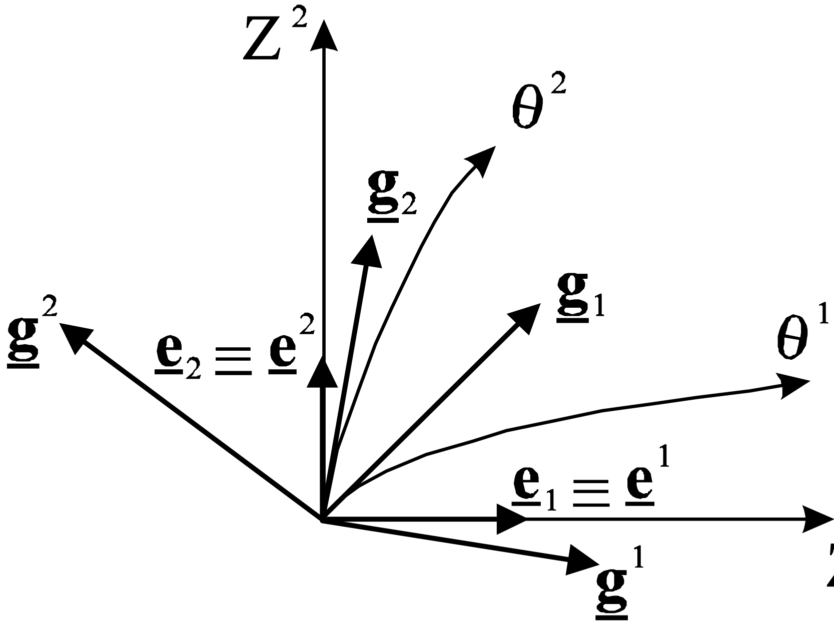

Fig. A.2. Covariant and contravariant base vectors

In Fig.A.2 we represent in a space $\Re^2$, at a point P, the covariant and contravariant base vectors, in order to provide the reader with a useful geometrical insight.

> It is important to note that in a Cartesian system the covariant and contravariant base vectors are coincident.

A.3 Metric of a coordinates system

If a position vector $\underline{\mathbf{r}}$ defines a point P in $\Re^3$ and a position vector $(\underline{\mathbf{r}} + \mathrm{d}\underline{\mathbf{r}})$ defines a neighboring point P', the distance between these points is given by

$$\mathrm{d}s \;=\; \sqrt{\mathrm{d}\underline{\mathbf{r}} \;\cdot\; \mathrm{d}\underline{\mathbf{r}}}\;. \tag{A.12}$$

A.3.1 Cartesian coordinates

In a Cartesian system in $\Re^3$, we have

$$\mathrm{d}s^2 \;=\; \mathrm{d}z^\alpha\,\mathrm{d}z^\beta\,(\underline{\mathbf{e}}_\alpha \;\cdot\; \underline{\mathbf{e}}_\beta)\;. \tag{A.13a}$$

If we call $\underline{\mathbf{e}}_\alpha \cdot \underline{\mathbf{e}}_\beta = \delta_{\alpha\beta}$, then

$$\mathrm{d}s^2 = \mathrm{d}z^\alpha \, \mathrm{d}z^\beta \, \delta_{\alpha\beta} \; . \tag{A.13b}$$

We call the nine number $\delta_{\alpha\beta}$ the *Cartesian components of the metric tensor* at P (this notation will be clarified in Sect. A.4.3).

From its definition, it is obvious that $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

A.3.2 Curvilinear coordinates. Covariant metric components

In an arbitrary curvilinear system $\{ \theta^i \; , \; i = 1, 2, 3 \}$ the distance between P and P' is given by

$$\mathrm{d}s^2 = \mathrm{d}\underline{\mathbf{r}} \cdot \mathrm{d}\underline{\mathbf{r}} = \mathrm{d}\theta^i \, \mathrm{d}\theta^j \, (\underline{\mathbf{g}}_i \cdot \underline{\mathbf{g}}_j) \; . \tag{A.14a}$$

We call

$$g_{ij} = g_{ji} = \underline{\mathbf{g}}_i \cdot \underline{\mathbf{g}}_j \tag{A.14b}$$

the covariant components of the metric tensor at P in the curvilinear system $\{\theta^i\}$ (this notation will be clarified in Section A.4.3.).

Using Eq. (A.8), we get

$$g_{ij} = \frac{\partial z^\alpha}{\partial \theta^i} \, \frac{\partial z^\beta}{\partial \theta^j} \, \delta_{\alpha\beta} \; . \tag{A.15a}$$

Defining a second curvilinear system $\{ \hat{\theta}_i \; , \; i = 1, 2, 3 \}$ and using Eq. (A.9b), we get

$$g_{ij} = \frac{\partial \hat{\theta}^l}{\partial \theta^i} \, \frac{\partial \hat{\theta}^m}{\partial \theta^j} \, \hat{g}_{lm} \; . \tag{A.15b}$$

Equations (A.15a-A.15b) are the reason for using the name "covariant" for the metric tensor components defined in Eq. (A.14b).

A.3.3 Curvilinear coordinates. Contravariant metric components

The scalars defined in Eq. (A.14b) by the dot product of the covariant base vectors were named covariant components of the metric tensor. In the same way, we define

$$g^{ij} = g^{ji} = \underline{\mathbf{g}}^i \cdot \underline{\mathbf{g}}^j \tag{A.16a}$$

the *contravariant components of the metric tensor at P*.

Using Eq. (A.11c), we get

$$g^{ij} = \frac{\partial \theta^i}{\partial \hat{\theta}^l} \, \frac{\partial \theta^j}{\partial \hat{\theta}^m} \, \hat{g}^{lm} \; . \tag{A.16b}$$

The above equation is the reason for using the name "contravariant" for the metric tensor components defined in Eq. (A.16a).

It is obvious that in the Cartesian system, we have

$$\delta^{\alpha\beta} = \delta_{\alpha\beta} . \tag{A.17a}$$

When $\{\hat{\theta}^i\}$ is a Cartesian system, Eq. (A.16b) is

$$g^{ij} = \frac{\partial\theta^i}{\partial z^\alpha} \frac{\partial\theta^j}{\partial z^\beta} \delta^{\alpha\beta} . \tag{A.17b}$$

A.3.4 Curvilinear coordinates. Mixed metric components

In any curvilinear coordinate system $\{\theta^i , i = 1, 2, 3 \}$ we can define the mixed components of the metric tensor as

$$g^i{}_j = \underline{\mathbf{g}}^i \cdot \underline{\mathbf{g}}_j = \delta^i_j \tag{A.18a}$$

$$g_j{}^i = \underline{\mathbf{g}}_j \cdot \underline{\mathbf{g}}^i = \delta^i_j . \tag{A.18b}$$

Example A.2. ◀◀◀◀◀

Any vector in $\Re^3$ can be written as a linear combination of the covariant base vectors; hence, we can write

$$\underline{\mathbf{g}}^i = \alpha^{ij} \underline{\mathbf{g}}_j .$$

When we postmultiply by $\underline{\mathbf{g}}^k$ on both sides, we obtain

$$\alpha^{ik} = \underline{\mathbf{g}}^i \cdot \underline{\mathbf{g}}^k = g^{ik} .$$

Thus, we have

$$\underline{\mathbf{g}}^i = g^{ij} \underline{\mathbf{g}}_j .$$

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Example A.3. ◀◀◀◀◀

Proceeding as we did in the previous examples, the reader can easily show that:

$$\underline{\mathbf{g}}_i = g_{ij} \underline{\mathbf{g}}^j .$$

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A.4 Tensors

We show in Sect. A.2 that in the space $\Re^3$ we can define two sets of linearly independent vectors: the covariant and the contravariant base vectors. Hence, any arbitrary vector in $\Re^3$ can be written as:

$$\underline{\mathbf{v}} \;=\; v^i\,\underline{\mathbf{g}}_i \;=\; v_i\,\underline{\mathbf{g}}^i\;. \tag{A.19}$$

We are now going to show that for the vector $\underline{\mathbf{v}}$ to remain *invariant* under coordinate changes, the components v^i should transform following a contravariant rule and the components v_i should transform following a covariant rule.

When we go from the system $\{\ \theta^i\ ,\ i=1,2,3\ \}$ to the system $\{\ \hat{\theta}^i\ ,\ i=1,2,3\ \}$, using Eqs. (A.9b), we obtain

$$\underline{\mathbf{v}} \;=\; v^i\,\underline{\mathbf{g}}_i \;=\; v^i\,\frac{\partial\hat{\theta}^j}{\partial\theta^i}\,\underline{\hat{\mathbf{g}}}_j \;=\; \hat{v}^j\,\underline{\hat{\mathbf{g}}}_j\;. \tag{A.20a}$$

Hence,

$$\hat{v}^j \;=\; v^i\,\frac{\partial\hat{\theta}^j}{\partial\theta^i} \qquad ;\quad j=1,2,3\;. \tag{A.20b}$$

We see from the above that when the coordinate system is changed, the components v^i transform following a contravariant rule.

Using Eq. (A.11c), we can write

$$\underline{\mathbf{v}} \;=\; \hat{v}_j\,\underline{\hat{\mathbf{g}}}^j \;=\; \hat{v}_j\,\frac{\partial\hat{\theta}^j}{\partial\theta^i}\,\underline{\mathbf{g}}^i \;=\; v_i\,\underline{\mathbf{g}}^i\;. \tag{A.21a}$$

Hence,

$$v_i \;=\; \hat{v}_j\,\frac{\partial\hat{\theta}^j}{\partial\theta^i} \qquad ;\quad i=1,2,3\;. \tag{A.21b}$$

We see from the above that when the coordinate system is changed, the components v^i transform following a covariant rule.

As a conclusion to this section, we can state that the invariance of $\underline{\mathbf{v}}$ under coordinate transformation requires the use of:

- *covariant components + contravariant base vectors*

or

- *contravariant components + covariant base vectors.*

A.4.1 Second-order tensors

Generalizing the concept of vectors that we presented above, we define as *second-order tensors* the following mathematical entities:

$$\underline{\underline{\mathbf{a}}} = a_{ij}\, \underline{\mathbf{g}}^i\, \underline{\mathbf{g}}^j = a^{ij}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j = a^i{}_j\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}^j = a_i{}^j\, \underline{\mathbf{g}}^i\, \underline{\mathbf{g}}_j \tag{A.22}$$

that remain *invariant under coordinate transformations.*

In the above equation, we used *tensorial or dyadic products between vectors* ($\underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j$; $\underline{\mathbf{g}}^i\, \underline{\mathbf{g}}^j$; $\underline{\mathbf{g}}^i\, \underline{\mathbf{g}}_j$; *etc.*) that we are going to formally define in this Section.

For the transformation $\{\theta^i\} \to \{\hat{\theta}^i\}$ using Eq. (A.9b), we get

$$\underline{\underline{\mathbf{a}}} = a^{ij}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j = a^{ij}\, \frac{\partial \hat{\theta}^k}{\partial \theta^i}\, \frac{\partial \hat{\theta}^l}{\partial \theta^j}\, \underline{\hat{\mathbf{g}}}_k\, \underline{\hat{\mathbf{g}}}_l = \hat{a}^{kl}\, \underline{\hat{\mathbf{g}}}_k\, \underline{\hat{\mathbf{g}}}_l\ . \tag{A.23a}$$

Thus, we have

$$\hat{a}^{kl} = a^{ij}\, \frac{\partial \hat{\theta}^k}{\partial \theta^i}\, \frac{\partial \hat{\theta}^l}{\partial \theta^j} \qquad ; \quad k,l = 1,2,3\ . \tag{A.23b}$$

That is to say, the components a^{ij} transform following a double contravariant rule.

In the same way, we can show that

$$\hat{a}_{kl} = a_{ij}\, \frac{\partial \theta^i}{\partial \hat{\theta}^k}\, \frac{\partial \theta^j}{\partial \hat{\theta}^l} \qquad ; \quad k,l = 1,2,3\ . \tag{A.23c}$$

That is to say, the components a_{ij} transform following a double covariant rule.

We can also show that:

$$\hat{a}^k{}_l = a^i{}_j\, \frac{\partial \hat{\theta}^k}{\partial \theta^i}\, \frac{\partial \theta^j}{\partial \hat{\theta}^l} \qquad ; \quad k,l = 1,2,3 \tag{A.23d}$$

$$\hat{a}_l{}^k = a_j{}^i\, \frac{\partial \theta^j}{\partial \hat{\theta}^l}\, \frac{\partial \hat{\theta}^k}{\partial \theta^i} \qquad ; \quad k,l = 1,2,3\ . \tag{A.23e}$$

That is to say, the components $a^i{}_j$ and $a_i{}^j$ transform following mixed rules.

From Eq. (A.22), we have

$$a_{kl}\, \underline{\mathbf{g}}^k\, \underline{\mathbf{g}}^l = a^{ij}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j\ . \tag{A.24a}$$

When we postmultiply by $\underline{\mathbf{g}}_r$ on both sides (the formal definition of this operation is given in what follows), we get

$$a_{kl}\, \underline{\mathbf{g}}^k\, \delta^l_r = a^{ij}\, g_{jr}\, \underline{\mathbf{g}}_i \tag{A.24b}$$

and if we now postmultiply (inner product) both sides by $\underline{\mathbf{g}}_s$, we obtain

$$a_{kl}\ \delta^l_r\ \delta^k_s\ =\ a^{ij}\ g_{is}\ g_{jr}\ . \qquad \text{(A.24c)}$$

Finally,

$$a_{sr}\ =\ a^{ij}\ g_{is}\ g_{jr} \qquad ;\quad r,s = 1,2,3\ . \qquad \text{(A.24d)}$$

In the same way, we can show the following relations for $k,l = 1,2,3$:

$$a_{kl}\ =\ a_k^{\ j}\ g_{jl} \qquad \text{(A.24e)}$$

$$a^{kl}\ =\ a_{ij}\ g^{ik}\ g^{jl}\ =\ a^k_{\ j}\ g^{jl} \qquad \text{(A.24f)}$$

$$a^k_{\ l}\ =\ a^{kj}\ g_{jl}\ =\ a_{jl}\ g^{jk}\ . \qquad \text{(A.24g)}$$

It is evident that we can raise and lower indices using the proper components of the metric tensor.

In Eq. (A.22) and the ones that followed, we wrote *dyads* of the type $\underline{\mathbf{g}}_i\ \underline{\mathbf{g}}_j$ or $\underline{\mathbf{g}}^i\ \underline{\mathbf{g}}^j$ or $\underline{\mathbf{g}}^i\ \underline{\mathbf{g}}_j$ or $\underline{\mathbf{g}}_i\ \underline{\mathbf{g}}^j$ that define an operation known as the *tensorial product* of two vectors.

To define the tensorial product of two vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ ($\underline{\mathbf{ab}}$ in our notation or $\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}$ in the notation used by other authors), we will define the properties of this new operation:

- Given a scalar α,

$$\alpha\ (\underline{\mathbf{ab}})\ =\ (\alpha\ \underline{\mathbf{a}})\underline{\mathbf{b}}\ =\ \underline{\mathbf{a}}(\alpha\ \underline{\mathbf{b}})\ =\ \alpha\underline{\mathbf{ab}}\ . \qquad \text{(A.25a)}$$

- Given a third vector $\underline{\mathbf{c}}$,

$$(\underline{\mathbf{ab}})\underline{\mathbf{c}}\ =\ \underline{\mathbf{a}}(\underline{\mathbf{bc}})\ =\ \underline{\mathbf{abc}} \qquad \text{(A.25b)}$$

and

$$\underline{\mathbf{a}}(\underline{\mathbf{b}}+\underline{\mathbf{c}})\ =\ \underline{\mathbf{ab}}\ +\underline{\mathbf{ac}}\ \neq\ \underline{\mathbf{ba}}\ +\ \underline{\mathbf{ca}} =\ (\underline{\mathbf{b}}+\underline{\mathbf{c}})\ \underline{\mathbf{a}}\ . \qquad \text{(A.25c)}$$

- In general,

$$\underline{\mathbf{ab}} \neq \underline{\mathbf{ba}}\ . \qquad \text{(A.25d)}$$

- The scalar product of a vector $\underline{\mathbf{c}}$ with the dyad $\underline{\mathbf{ab}}$ is a vector,

$$\underline{\mathbf{c}}\ \cdot\ (\underline{\mathbf{ab}})\ =\ (\underline{\mathbf{c}}\ \cdot\ \underline{\mathbf{a}})\underline{\mathbf{b}}\ , \qquad \text{(A.25e)}$$

where $(\underline{\mathbf{c}}\ \cdot\ \underline{\mathbf{a}})$ is a scalar. Also,

$$(\underline{\mathbf{ab}}) \cdot \underline{\mathbf{c}} = \underline{\mathbf{a}}(\underline{\mathbf{b}} \cdot \underline{\mathbf{c}}) \neq \underline{\mathbf{c}} \cdot (\underline{\mathbf{ab}}) = (\underline{\mathbf{c}} \cdot \underline{\mathbf{a}})\underline{\mathbf{b}} \, . \tag{A.25f}$$

It should be notd that $(\underline{\mathbf{ab}}) \cdot \underline{\mathbf{c}}$ is a vector with the direction of the vector $\underline{\mathbf{a}}$, while $\underline{\mathbf{c}} \cdot (\underline{\mathbf{ab}})$ is a vector with the direction of the vector $\underline{\mathbf{b}}$.

- The scalar or inner product between two dyads is another dyad:

$$(\underline{\mathbf{ab}}) \cdot (\underline{\mathbf{cd}}) = (\underline{\mathbf{b}} \cdot \underline{\mathbf{c}}) \, \underline{\mathbf{ad}} \, . \tag{A.25g}$$

- The double scalar or inner product between two dyads is a scalar:

$$(\underline{\mathbf{ab}}) : (\underline{\mathbf{cd}}) = (\underline{\mathbf{a}} \cdot \underline{\mathbf{c}}) \, (\underline{\mathbf{b}} \cdot \underline{\mathbf{d}}) \, . \tag{A.25h}$$

Besides (Malvern 1969),

$$(\underline{\mathbf{ab}}) \cdot\cdot (\underline{\mathbf{cd}}) = (\underline{\mathbf{a}} \cdot \underline{\mathbf{d}}) \, (\underline{\mathbf{b}} \cdot \underline{\mathbf{c}}) \tag{A.25i}$$

which is a scalar too.

Using the above definition, we can perform the scalar product of the second-order tensor $\underline{\underline{\mathbf{a}}}$ defined by Eq. (A.22) and the vector $\underline{\mathbf{v}}$ defined by Eq. (A.19),

$$\underline{\underline{\mathbf{a}}} \cdot \underline{\mathbf{v}} = a_{ik} \, v^k \, \underline{\mathbf{g}}^i = a^{ik} \, v_k \, \underline{\mathbf{g}}_i = a^i{}_k \, v^k \, \underline{\mathbf{g}}_i = a_i{}^k \, v_k \, \underline{\mathbf{g}}^i \tag{A.26a}$$

then, we obtain a vector $\underline{\mathbf{b}} = \underline{\underline{\mathbf{a}}} \cdot \underline{\mathbf{v}}$ with:

- covariant components: $b_i = a_{ik} \, v^k = a_i{}^k \, v_k$
- contravariant components: $b^i = a^{ik} \, v_k = a^i{}_k \, v^k$.

It is easy to show that $(\underline{\mathbf{v}} \cdot \underline{\underline{\mathbf{a}}})$ is also a vector and that in general

$$\underline{\mathbf{v}} \cdot \underline{\underline{\mathbf{a}}} \neq \underline{\underline{\mathbf{a}}} \cdot \underline{\mathbf{v}} \, . \tag{A.26b}$$

Eigenvalues and eigenvectors of second-order tensors

We say that a vector $\underline{\mathbf{v}}$ is an *eigenvector* of a second-order tensor $\underline{\underline{\mathbf{a}}}$ if

$$\underline{\underline{\mathbf{a}}} \cdot \underline{\mathbf{v}} = \lambda \, \underline{\mathbf{v}} \tag{A.27}$$

and we call λ the *eigenvalue* associated to the eigendirection $\underline{\mathbf{v}}$.

It is easy to show that the following relation holds

$$(\, a_{ij} - \lambda \, g_{ij} \,) \, v^j = 0 \, . \tag{A.28}$$

Equation (A.28) represents a system of 3 homogeneous equations $(j = 1, 2, 3)$ with 3 unknowns (v^1, v^2, v^3) . To obtain a solution different from the trivial one, we must have

$$| \, a_{ij} \; - \; \lambda \, g_{ij} \, | \; = \; 0 \; . \tag{A.29}$$

The above is a cubic equation in λ that leads to 3 eigenvalues and therefore 3 associated eigendirections. It is obvious that if a pair $(\lambda, \underline{\mathbf{v}})$ satisfies Eq. (A.27), the pair $(\lambda, \alpha \; \underline{\mathbf{v}})$ will also do so. Hence, the modulus of the eigenvectors remains undefined.

The following properties can be derived:

- If $\underline{\underline{\mathbf{a}}}$ is symmetric, the eigenvalues and eigenvectors are real.

Proof. (Green & Zerna 1968)

Assume λ is not real, then

$$\lambda \; = \; \alpha \; + \; i \, \beta \tag{A.30a}$$

$$v^j \; = \; \eta^j \; + \; i \, \mu^j \; . \tag{A.30b}$$

From Eq. (A.28), equating real and imaginary parts,

$$(a_{ij} \; - \; \alpha \, g_{ij}) \, \eta^j \; + \; \beta \, g_{ij} \, \mu^j \; = \; 0 \tag{A.30c}$$

$$(a_{ij} \; - \; \alpha \, g_{ij}) \, \mu^j \; - \; \beta \, g_{ij} \, \eta^j \; = \; 0 \; . \tag{A.30d}$$

After some algebra, from the two above equations, we get

$$\beta \, (g_{ij} \, \eta^i \, \eta^j \; + \; g_{ij} \, \mu^i \, \mu^j) \; = \; 0 \tag{A.30e}$$

for $a_{ij} \; = \; a_{ji}$.

Since all the η^i and μ^i cannot be zero and the terms $(g_{ij} \, \eta^i \, \eta^j)$ and $(g_{ij} \, \mu^i \, \mu^j)$ are always positive (see Eq. (A.14a)), then

$$\beta \; = \; 0 \; . \tag{A.30f}$$

Therefore, the eigenvalues are always real and to satisfy Eq. (A.28) the eigenvectors shall also be real.

- If $\underline{\underline{\mathbf{a}}}$ is symmetric, the eigenvectors form an orthogonal set.

Proof. (Crandall 1956)

For two pairs $(\lambda_1, \underline{\mathbf{v}}_1)$ and $(\lambda_2, \underline{\mathbf{v}}_2)$ from Eq. (A.27)

$$\underline{\mathbf{v}}_1 \cdot \underline{\underline{\mathbf{a}}} \cdot \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_2 \cdot \underline{\underline{\mathbf{a}}} \cdot \underline{\mathbf{v}}_1 = (\lambda_2 - \lambda_1)\, \underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2 \,. \tag{A.31}$$

For a symmetric tensor, the l.h.s. of the above equation is zero. Hence,
∘ If $\lambda_1 \neq \lambda_2$ then $\underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2 = 0$. That is to say, $\underline{\mathbf{v}}_1$ and $\underline{\mathbf{v}}_2$ are orthogonal.
∘ If $\lambda_1 = \lambda_2$ there are infinite vectors $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$ that satisfy the above equation. Among them we can select a pair of orthogonal vectors.

Hence, in general we assess that for symmetric second-order tensors, the eigenvectors are orthogonal.

Example A.4. ◀◀◀◀◀
As $\underline{\underline{\mathbf{a}}}$ is a symmetric second order tensor, with eigenvalues λ_I and eigenvectors $\underline{\mathbf{v}}_I$ $(I = 1, 2, 3)$ with $|\underline{\mathbf{v}}_I| = 1$, the *canonical form* of $\underline{\underline{\mathbf{a}}}$ is:

$$\underline{\underline{\mathbf{a}}} = \lambda_1\, \underline{\mathbf{v}}_1\, \underline{\mathbf{v}}_1 + \lambda_2\, \underline{\mathbf{v}}_2\, \underline{\mathbf{v}}_2 + \lambda_3\, \underline{\mathbf{v}}_3\, \underline{\mathbf{v}}_3$$

also known as the *diagonalized form.* ◀◀◀◀◀

A.4.2 n-order tensors

In the same way that we defined the tensorial products of two vectors (dyad), we can define the tensorial product of n vectors (*n-poliad*). Therefore, we can define mathematical entities of the type:

$$\begin{aligned} \mathbf{t} &= t^{ij\ldots n}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_n = t_{ij\ldots n}\, \underline{\mathbf{g}}^i\, \underline{\mathbf{g}}^j \cdots \underline{\mathbf{g}}^n \\ &= t^{ij\ldots k}{}_{lm\ldots n}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^l\, \underline{\mathbf{g}}^m \cdots \underline{\mathbf{g}}^n \end{aligned} \tag{A.32}$$

which we call *tensors of order n* and we associate to them the property of remaining *invariant when coordinate transformations are performed.*

When we go from the curvilinear system $\{\, \theta^i \,,\ i = 1, 2, 3 \,\}$ to the curvilinear system $\{\, \hat{\theta}^i \,,\ i = 1, 2, 3 \,\}$, *due to the invariance property*, we get

$$\begin{aligned} \mathbf{t} &= t^{ij\ldots k}{}_{lm\ldots n}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^l\, \underline{\mathbf{g}}^m \cdots \underline{\mathbf{g}}^n \\ &= \hat{t}^{ab\ldots c}{}_{de\ldots f}\, \underline{\hat{\mathbf{g}}}_a\, \underline{\hat{\mathbf{g}}}_b \cdots \underline{\hat{\mathbf{g}}}_c\, \underline{\hat{\mathbf{g}}}^d\, \underline{\hat{\mathbf{g}}}^e \cdots \underline{\hat{\mathbf{g}}}^f \,. \end{aligned} \tag{A.33a}$$

Hence, the following relations can be derived:

$$\hat{t}^{ab\ldots c}{}_{de\ldots f} \;=\; t^{ij\ldots k}{}_{lm\ldots n}\;(\hat{\underline{\mathbf{g}}}^{a}\;\cdot\;\underline{\mathbf{g}}_{i})\;(\hat{\underline{\mathbf{g}}}^{b}\;\cdot\;\underline{\mathbf{g}}_{j})\;\cdots\;(\hat{\underline{\mathbf{g}}}^{c}\;\cdot\;\underline{\mathbf{g}}_{k})$$
$$(\hat{\underline{\mathbf{g}}}_{d}\;\cdot\;\underline{\mathbf{g}}^{l})\;(\hat{\underline{\mathbf{g}}}_{e}\;\cdot\;\underline{\mathbf{g}}^{m})\;\cdots\;(\hat{\underline{\mathbf{g}}}_{f}\;\cdot\;\underline{\mathbf{g}}^{n}) \tag{A.33b}$$

and using Eqs. (A.9b) and (A.11c), we obtain

$$\hat{t}^{ab\ldots c}{}_{de\ldots f} \;=\; t^{ij\ldots k}{}_{lm\ldots n}\;\frac{\partial\hat{\theta}^{a}}{\partial\theta^{i}}\;\frac{\partial\hat{\theta}^{b}}{\partial\theta^{j}}\;\cdots\;\frac{\partial\hat{\theta}^{c}}{\partial\theta^{k}}\;\frac{\partial\theta^{l}}{\partial\hat{\theta}^{d}}\;\frac{\partial\theta^{m}}{\partial\hat{\theta}^{e}}\;\cdots\;\frac{\partial\theta^{n}}{\partial\hat{\theta}^{f}}\;. \tag{A.33c}$$

A.4.3 The metric tensor

As a particular but important example of second-order tensors, we will refer in this section to the *metric tensor*, $\underline{\underline{\mathbf{g}}}$,

$$\underline{\underline{\mathbf{g}}} \;=\; g_{ij}\;\underline{\mathbf{g}}^{i}\;\underline{\mathbf{g}}^{j} \;=\; g^{ij}\;\underline{\mathbf{g}}_{i}\;\underline{\mathbf{g}}_{j} \;=\; \delta^{i}_{j}\;\underline{\mathbf{g}}_{i}\;\underline{\mathbf{g}}^{j} \;=\; \delta^{j}_{i}\;\underline{\mathbf{g}}^{i}\;\underline{\mathbf{g}}_{j}\;. \tag{A.34}$$

In Sects. A.3.1 - A.3.3, we introduced the covariant, contravariant and mixed components of this tensor.

We can rewrite Eq.(A.14a) as:

$$ds^{2} \;=\; d\underline{\mathbf{r}}\;\cdot\;\underline{\underline{\mathbf{g}}}\;\cdot\;d\underline{\mathbf{r}} \;=\; \left(d\theta^{i}\;\underline{\mathbf{g}}_{i}\right)\;\cdot\;\left(g_{kl}\;\underline{\mathbf{g}}^{k}\;\underline{\mathbf{g}}^{l}\right)\;\cdot\;\left(d\theta^{j}\;\underline{\mathbf{g}}_{j}\right) \tag{A.35a}$$

and therefore,

$$ds^{2} \;=\; d\theta^{i}\;\;d\theta^{j}\;g_{kl}\;\delta^{k}_{i}\;\delta^{l}_{j} \;=\; d\theta^{i}\;\;d\theta^{j}\;g_{ij}\;. \tag{A.35b}$$

Going back to Eq. (A.34), we post-multiply both sides by the vector $\underline{\mathbf{g}}^{p}$ and

$$g_{ij}\;\underline{\mathbf{g}}^{i}\;\underline{\mathbf{g}}^{j}\;\cdot\;\underline{\mathbf{g}}^{p} \;=\; g^{kl}\;\underline{\mathbf{g}}_{k}\;\underline{\mathbf{g}}_{l}\;\cdot\;\underline{\mathbf{g}}^{p}\;. \tag{A.36a}$$

Operating, we get

$$g_{ij}\;g^{jp}\;\underline{\mathbf{g}}^{i} \;=\; g^{kl}\;\delta^{p}_{l}\;\underline{\mathbf{g}}_{k}\;. \tag{A.36b}$$

Using Eqs. (A.15a) and (A.17b), we arrive at

$$g_{ij}\;g^{jp} \;=\; \frac{\partial z^{\alpha}}{\partial\theta^{i}}\;\frac{\partial z^{\beta}}{\partial\theta^{j}}\;\delta_{\alpha\beta}\;\frac{\partial\theta^{j}}{\partial z^{\gamma}}\;\frac{\partial\theta^{p}}{\partial z^{\delta}}\;\delta^{\gamma\delta} \;=\; \frac{\partial z^{\alpha}}{\partial\theta^{i}}\;\frac{\partial z^{\alpha}}{\partial z^{\gamma}}\;\frac{\partial\theta^{p}}{\partial z^{\gamma}}\;. \tag{A.36c}$$

Rearranging,

$$g_{ij}\;g^{jp} \;=\; \frac{\partial\theta^{p}}{\partial z^{\alpha}}\;\frac{\partial z^{\alpha}}{\partial\theta^{i}} \;=\; \frac{\partial\theta^{p}}{\partial\theta^{i}} \;=\; \delta^{p}_{i}\;. \tag{A.36d}$$

Using the above in Eq. (A.36b), we finally obtain

$$\underline{\mathbf{g}}^{p} \;=\; g^{pk}\;\underline{\mathbf{g}}_{k}\;. \tag{A.36e}$$

The above equality was also derived in Example A.2. In an identical way, we can also derive the result of the Example A.3.

A.4.4 The Levi-Civita tensor

The *Cartesian components of the Levi-Civita or permutation tensor* are defined as:

$$e_{\alpha\beta\gamma} \;=\; e^{\alpha\beta\gamma} \;=\left\{ \begin{array}{c} 0 \text{ when two of the indices are equal} \\ 1 \text{ when the indices are arranged as 1,2,3} \\ -1 \text{ when the indices are arranged as 1,3,2} \end{array} \right\} .$$

By using the tensorial components transformation rules in an arbitrary curvilinear system $\{ \; \theta^i \quad , i = 1, 2, 3 \; \}$ and for the covariant components, we get

$$\epsilon_{ijk} \;=\; \frac{\partial z^\alpha}{\partial \theta^i} \, \frac{\partial z^\beta}{\partial \theta^j} \, \frac{\partial z^\gamma}{\partial \theta^k} \, e_{\alpha\beta\gamma} \; . \tag{A.37a}$$

Taking into account that the determinant of a (3×3) matrix can be written as:

$$\left| a^i{}_j \right| \;=\; e_{rst} \; a^r{}_1 \; a^s{}_2 \; a^t{}_3 \tag{A.37b}$$

it is easy to show that

$$e_{ijk} \; |a^m{}_n| \;=\; e_{rst} \; a^r{}_i \; a^s{}_j \; a^t{}_k \; . \tag{A.37c}$$

Another important relation is

$$e_{\alpha\beta\gamma} \; e^{\alpha\delta\epsilon} \;=\; \delta^\delta_\beta \; \delta^\epsilon_\gamma \;-\; \delta^\epsilon_\beta \; \delta^\delta_\gamma \; . \tag{A.37d}$$

Introducing the above relation in Eq. (A.37a) and also using Eq. (A.2), we get

$$\epsilon_{ijk} \;=\; e_{ijk} \; \left| \frac{\partial z^m}{\partial \theta^n} \right| \; . \tag{A.37e}$$

In the same way, we can show that

$$\epsilon^{ijk} \;=\; e^{ijk} \; \left| \frac{\partial \theta^n}{\partial z^m} \right| \; . \tag{A.38}$$

Some authors define the components of the permutation tensor in any curvilinear system using the same definition that we just used for the Cartesian components. In this way, the tensorial components transformation rules are not fulfilled (invariance is lost) and the permutation tensor in this case is called a *pseudotensor.*

In a Cartesian system, we define the *cross product between two vectors* as

$$\underline{\mathbf{e}}_\alpha \;\times\; \underline{\mathbf{e}}_\beta \;=\; e_{\alpha\beta\gamma} \; \underline{\mathbf{e}}^\gamma \; . \tag{A.39}$$

Taking into account that

$$\underline{\mathbf{e}}_{\alpha} = \frac{\partial \theta^l}{\partial z^{\alpha}} \, \underline{\mathbf{g}}_l \tag{A.40a}$$

$$\underline{\mathbf{e}}^{\gamma} = \frac{\partial z^{\gamma}}{\partial \theta^n} \, \underline{\mathbf{g}}^n \tag{A.40b}$$

and using the above-derived curvilinear components of the Levi-Civita tensor, we get

$$\frac{\partial \theta^l}{\partial z^{\alpha}} \frac{\partial \theta^m}{\partial z^{\beta}} \left[\underline{\mathbf{g}}_l \times \underline{\mathbf{g}}_m - \epsilon_{lmn} \, \underline{\mathbf{g}}^n \right] = \underline{\mathbf{0}} \, . \tag{A.40c}$$

Considering that the above expression is valid in any curvilinear system, we obtain

$$\underline{\mathbf{g}}_l \times \underline{\mathbf{g}}_m = \epsilon_{lmn} \, \underline{\mathbf{g}}^n \, . \tag{A.41}$$

Example A.5. ◀◀◀◀◀

For a second-order tensor we can write Eq. (A.29) using mixed components as:

$$| \, a^i{}_j - \lambda \, \delta^i{}_j \, | = 0 \, .$$

Hence, using Eq. (A.37b)

$$e_{rst} \left[(a^r{}_1 - \lambda \, \delta^r{}_1) \, (a^s{}_2 - \lambda \, \delta^s{}_2) \, (a^t{}_3 - \lambda \, \delta^t{}_3) \right] = 0 \, .$$

After some algebra (Flügge 1972) we get the *characteristic equation* of the second-order tensor $\underline{\underline{\mathbf{a}}}$.

$$\lambda^3 - a^i{}_i \, \lambda^2 + \frac{1}{2} \, (a^i{}_i \, a^j{}_j - a^i{}_j \, a^j{}_i) \, \lambda - |a^i{}_j| = 0 \, .$$

Since the eigenvectors of $\underline{\underline{\mathbf{a}}}$ are independent of the coordinate system we use to describe the tensor, the coefficients of the above equation are *invariant* against coordinate transformations. We define the invariants as:

$$\begin{aligned}
I_I &= a^i{}_i = a^{ij} \, g_{ij} = a_{ij} \, g^{ij} = \underline{\underline{\mathbf{a}}} : \underline{\underline{\mathbf{g}}} \, , \\
I_{II} &= \frac{1}{2} \left[a^i{}_j \, a^j{}_i - a^i{}_i \, a^j{}_j \right] = \frac{1}{2} \left(\underline{\underline{\mathbf{a}}} \cdot\cdot \, \underline{\underline{\mathbf{a}}} - {I_I}^2 \right) \, , \\
I_{III} &= |a^i{}_j| \, ,
\end{aligned}$$

Finally, the characteristic equation can be written as:

$$\lambda^3 - I_I \, \lambda^2 - I_{II} \, \lambda - I_{III} = 0 \, .$$

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Example A.6. ◀◀◀◀◀

In a plane normal to the axis z^3, we define an arbitrary curvilinear system $\{ \theta^i \ , \ i = 1, 2 \}$ with $\theta^3 = z^3$. The base vectors of the curvilinear system are

$$\begin{aligned} \underline{\mathbf{g}}_1 &= \frac{\partial z^1}{\partial \theta^1}\, \underline{\mathbf{e}}_1 + \frac{\partial z^2}{\partial \theta^1}\, \underline{\mathbf{e}}_2 \ , \\ \underline{\mathbf{g}}_2 &= \frac{\partial z^1}{\partial \theta^2}\, \underline{\mathbf{e}}_1 + \frac{\partial z^2}{\partial \theta^2}\, \underline{\mathbf{e}}_2 \ , \\ \underline{\mathbf{g}}_3 &= \underline{\mathbf{e}}_3 \ . \end{aligned}$$

We define the area differential in the plane normal to z^3 as:

$$\mathrm{d}\underline{\mathbf{A}}_3 = \mathrm{d}\theta^1\, \mathrm{d}\theta^2\, \underline{\mathbf{g}}_1 \times \underline{\mathbf{g}}_2$$

and after some algebra, we get

$$\mathrm{d}\underline{\mathbf{A}}_3 = |J_3|\ \mathrm{d}\theta^1\, \mathrm{d}\theta^2\, \underline{\mathbf{e}}_3$$

where $|J_3| = \left(\frac{\partial z^1}{\partial \theta^1} \frac{\partial z^2}{\partial \theta^2} - \frac{\partial z^2}{\partial \theta^1} \frac{\partial z^1}{\partial \theta^2} \right)$. In the particular case when the curvilinear system $\{ \theta^i \}$ is in fact a Cartesian system $\{ \hat{z}^i \}$, it is easy to show that $|J_3| = 1$. Therefore,

$$\mathrm{d}\underline{\mathbf{A}}_3 = \mathrm{d}\hat{z}^1\, \mathrm{d}\hat{z}^2\, \underline{\mathbf{e}}_3 \ .$$

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Example A.7. ◀◀◀◀◀

We define in the $\Re^3$ space an arbitrary curvilinear system $\{ \theta^i \ , \ i = 1, 2, 3 \}$ with the following covariant base vectors

$$\underline{\mathbf{g}}_i = \frac{\partial z^\alpha}{\partial \theta^i}\, \underline{\mathbf{e}}_\alpha \ .$$

We also define

$$\mathrm{d}V = \mathrm{d}\theta^1\, \mathrm{d}\theta^2\, \mathrm{d}\theta^3 \left[\underline{\mathbf{g}}_1 \cdot (\underline{\mathbf{g}}_2 \times \underline{\mathbf{g}}_3) \right]$$

and after some algebra, we get

$$\mathrm{d}V = \left| \frac{\partial z^m}{\partial \theta^n} \right| \mathrm{d}\theta^1\, \mathrm{d}\theta^2\, \mathrm{d}\theta^3 \ .$$

In the particular case when the curvilinear system $\{ \theta^i \}$ is in fact a Cartesian system $\{ \hat{z}^i \}$, it is easy to show that, $\left| \frac{\partial z^m}{\partial \theta^n} \right| = 1$. Therefore,

$$\mathrm{d}V = \mathrm{d}\hat{z}^1\, \mathrm{d}\hat{z}^2\, \mathrm{d}\hat{z}^3 \ .$$

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A.5 The quotient rule

Let $A_{(ijkrs)}$ be a set of 243 quantities, B^{rs} the contravariant components of an arbitrary second-order tensor (independent of $A_{(ijkrs)}$) and D_{ijk} the covariant components of a third-order tensor.

If, in any coordinate system, the relation

$$D_{ijk} \;=\; A_{(ijkrs)}\; B^{rs} \tag{A.42}$$

is satisfied, then we are going to prove that the $A_{(ijkrs)}$ are the covariant components of a fifth order tensor.

Proof. Since $\underline{\underline{\underline{\mathbf{D}}}}$ and $\underline{\underline{\mathbf{B}}}$ *are tensors*, when we change the coordinate system from $\{\ \theta^i\ ,\ i=1,2,3\ \}$ to $\{\ \hat{\theta}^i\ ,\ i=1,2,3\ \}$, we get

$$\hat{D}_{lmn} \;=\; \frac{\partial \theta^i}{\partial \hat{\theta}^l}\,\frac{\partial \theta^j}{\partial \hat{\theta}^m}\,\frac{\partial \theta^k}{\partial \hat{\theta}^n}\; D_{ijk} \tag{A.43a}$$

$$B^{rs} \;=\; \frac{\partial \theta^r}{\partial \hat{\theta}^p}\,\frac{\partial \theta^s}{\partial \hat{\theta}^q}\; \hat{B}^{pq}\ . \tag{A.43b}$$

Using Eqs. (A.42) and (A.43b) in Eq. (A.43a), we get

$$\hat{D}_{lmn} \;=\; \frac{\partial \theta^i}{\partial \hat{\theta}^l}\,\frac{\partial \theta^j}{\partial \hat{\theta}^m}\,\frac{\partial \theta^k}{\partial \hat{\theta}^n}\; A_{(ijkrs)}\; \frac{\partial \theta^r}{\partial \hat{\theta}^p}\,\frac{\partial \theta^s}{\partial \hat{\theta}^q}\; \hat{B}^{pq}\ . \tag{A.44a}$$

We can write

$$\hat{D}_{lmn} \;=\; \hat{A}_{(lmnpq)}\; \hat{B}^{pq} \tag{A.44b}$$

substracting Eq. (A.44a) from Eq. (A.44b), we get

$$\left[\ \hat{A}_{(lmnpq)} \;-\; \frac{\partial \theta^i}{\partial \hat{\theta}^l}\,\frac{\partial \theta^j}{\partial \hat{\theta}^m}\,\frac{\partial \theta^k}{\partial \hat{\theta}^n}\,\frac{\partial \theta^r}{\partial \hat{\theta}^p}\,\frac{\partial \theta^s}{\partial \hat{\theta}^q}\; A_{(ijkrs)}\ \right] \hat{B}^{pq} \;=\; 0\ . \tag{A.44c}$$

Since $\underline{\underline{\mathbf{B}}}$ is an arbitrary second order tensor, from the above equation, we obtain the following relation:

$$\hat{A}_{(lmnpq)} \;=\; \frac{\partial \theta^i}{\partial \hat{\theta}^l}\,\frac{\partial \theta^j}{\partial \hat{\theta}^m}\,\frac{\partial \theta^k}{\partial \hat{\theta}^n}\,\frac{\partial \theta^r}{\partial \hat{\theta}^p}\,\frac{\partial \theta^s}{\partial \hat{\theta}^q}\; A_{(ijkrs)}\ . \tag{A.44d}$$

The above equation shows that the $A_{(ijkrs)}$ transform according to a covariant transformation rule that shows they are the covariant components of a fifth-order tensor.

The generalization of the case that we analyzed, the quotient rule, is a tool for identifying general tensors.

Example A.8. ◀◀◀◀◀

Let us consider the vectors (first-order tensors)

$$\begin{aligned} \underline{\mathbf{x}} &= x^r \, \underline{\mathbf{g}}_r \\ \underline{\mathbf{y}} &= y^r \, \underline{\mathbf{g}}_r \\ \underline{\mathbf{z}} &= z_r \, \underline{\mathbf{g}}^r \, . \end{aligned}$$

If we know that

$$\alpha = A^r{}_{st} \, x^s \, y^t \, z_r$$

is invariant under coordinate transformations (a scalar), then the quotient rule indicates that the $A^r{}_{st}$ are the mixed components of the following tensor

$$\underline{\underline{\underline{\mathbf{A}}}} = A^r{}_{st} \, \underline{\mathbf{g}}_r \, \underline{\mathbf{g}}^s \, \underline{\mathbf{g}}^t \, .$$

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A.6 Covariant derivatives

A.6.1 Covariant derivatives of a vector

Contravariant components

Given a vector $\underline{\mathbf{v}}$, we can define it using its Cartesian components as

$$\underline{\mathbf{v}} = v^\alpha \, \underline{\mathbf{e}}_\alpha \, , \tag{A.45a}$$

and since the base vectors of a Cartesian system do not change with the coordinates, we get

$$\frac{\partial \underline{\mathbf{v}}}{\partial z^\beta} = \frac{\partial v^\alpha}{\partial z^\beta} \, \underline{\mathbf{e}}_\alpha \, . \tag{A.45b}$$

Using, in the *Euclidean space*, a system of arbitrary curvilinear coordinates $\{\theta^i, i = 1, 2, 3, \}$, we get

$$\underline{\mathbf{v}} = v^s \, \underline{\mathbf{g}}_s \tag{A.46a}$$

$$\frac{\partial \underline{\mathbf{v}}}{\partial \theta^n} = \frac{\partial v^s}{\partial \theta^n} \, \underline{\mathbf{g}}_s + v^s \, \frac{\partial \underline{\mathbf{g}}_s}{\partial \theta^n} \, . \tag{A.46b}$$

Using Eq. (A.8), we obtain

$$\frac{\partial \underline{\mathbf{g}}_s}{\partial \theta^n} = \frac{\partial^2 z^\alpha}{\partial \theta^s \, \partial \theta^n} \, \underline{\mathbf{e}}_\alpha \tag{A.46c}$$

and using it once more,

$$\frac{\partial \underline{\mathbf{g}}_s}{\partial \theta^n} = \frac{\partial^2 z^\alpha}{\partial \theta^s \, \partial \theta^n} \, \frac{\partial \theta^p}{\partial z^\alpha} \, \underline{\mathbf{g}}_p = \mathit{\Gamma}^p_{sn} \, \underline{\mathbf{g}}_p \; . \tag{A.46d}$$

$\mathit{\Gamma}^p_{sn}$ is defined as the *Christoffel symbol of the second kind in the Euclidean space*:

$$\mathit{\Gamma}^p_{sn} = \frac{\partial^2 z^\alpha}{\partial \theta^s \, \partial \theta^n} \, \frac{\partial \theta^p}{\partial z^\alpha} \; . \tag{A.47}$$

It should be noted that:

- The Christoffel symbol of the second kind is a function of the coordinate system under consideration $\{ \theta^i \}$ and of the coordinates of the point where the calculations are performed.
- The Christoffel symbols of the second kind are not tensorial components and therefore do not transform as such,

$$\hat{\mathit{\Gamma}}^a_{bc} = \frac{\partial^2 z^\alpha}{\partial \hat{\theta}^b \, \partial \hat{\theta}^c} \, \frac{\partial \hat{\theta}^a}{\partial z^\alpha} \; . \tag{A.48a}$$

In general,

$$\hat{\mathit{\Gamma}}^a_{bc} \neq \frac{\partial \hat{\theta}^a}{\partial \theta^p} \, \frac{\partial \theta^s}{\partial \hat{\theta}^b} \, \frac{\partial \theta^n}{\partial \hat{\theta}^c} \; \mathit{\Gamma}^p_{sn} \; . \tag{A.48b}$$

It is obvious from Eq. (A.47) that

$$\mathit{\Gamma}^p_{sn} = \mathit{\Gamma}^p_{ns} \; . \tag{A.49}$$

It is important to note that in general Eq. (A.49) *is not necessarily valid in a non-Euclidean space.*

- In the Cartesian coordinate system $\mathit{\Gamma}^\alpha_{\beta\gamma} = 0$.

From Eqs. (A.46b), (A.46d) and (A.47), we get

$$\frac{\partial \underline{\mathbf{v}}}{\partial \theta^n} = \left[\frac{\partial v^p}{\partial \theta^n} + \mathit{\Gamma}^p_{sn} \, v^s \right] \underline{\mathbf{g}}_p \; . \tag{A.50a}$$

Defining

$$v^p|_n = \frac{\partial v^p}{\partial \theta^n} + \mathit{\Gamma}^p_{sn} \, v^s \tag{A.50b}$$

we can write

$$\frac{\partial \underline{\mathbf{v}}}{\partial \theta^n} = v^p|_n \, \underline{\mathbf{g}}_p \,. \tag{A.50c}$$

We call $v^p|_n$ *the covariant derivative of the contravariant components of* $\underline{\mathbf{v}}$.

We are going to show in Sect. A.7. that the $v^p|_n$ are mixed components of a second-order tensor and that the subindex n, associated to the variable θ^n, transforms in a covariant way.

Example A.9. ◀◀◀◀◀

Since

$$g_{ij} = \underline{\mathbf{g}}_i \cdot \underline{\mathbf{g}}_j$$

and using Eq. (A.46d), we get

$$\frac{\partial g_{ij}}{\partial \theta^l} = \Gamma^p_{il} \, g_{pj} + \Gamma^p_{jl} \, g_{ip} \,.$$

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Example A.10. ◀◀◀◀◀

From the above result, we get

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \theta^l} + \frac{\partial g_{jl}}{\partial \theta^i} - \frac{\partial g_{li}}{\partial \theta^j} &= \Gamma^p_{il} \, g_{pj} + \Gamma^p_{jl} \, g_{pi} + \Gamma^p_{ji} \, g_{pl} \\
&+ \Gamma^p_{li} \, g_{pj} - \Gamma^p_{lj} \, g_{pi} - \Gamma^p_{ij} \, g_{pl} \\
&= (\Gamma^p_{il} + \Gamma^p_{li}) \, g_{pj} + \left(\Gamma^p_{jl} - \Gamma^p_{lj}\right) g_{pi} \\
&+ \left(\Gamma^p_{ji} - \Gamma^p_{ij}\right) g_{pl} \,.
\end{aligned}$$

In the Euclidean space,

$$g_{pj} \, \Gamma^p_{il} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial \theta^l} + \frac{\partial g_{jl}}{\partial \theta^i} - \frac{\partial g_{li}}{\partial \theta^j} \right) .$$

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Covariant components

We are now going to perform the derivations of the previous Section but, in the present case, for a vector defined using its covariant components and contravariant base vectors, the following results

$$\frac{\partial \underline{\mathbf{v}}}{\partial \theta^n} = \frac{\partial v_s}{\partial \theta^n}\, \underline{\mathbf{g}}^s + v_s\, \frac{\partial \underline{\mathbf{g}}^s}{\partial \theta^n} \,. \tag{A.51}$$

Taking into account that $\underline{\mathbf{g}}^s \cdot \underline{\mathbf{g}}_t = \delta^s_t$, we get

$$\frac{\partial \underline{\mathbf{g}}^s}{\partial \theta^n} \cdot \underline{\mathbf{g}}_t + \underline{\mathbf{g}}^s \cdot \frac{\partial \underline{\mathbf{g}}_t}{\partial \theta^n} = 0 \,. \tag{A.52a}$$

Using Eq. (A.46d) in the above,

$$\frac{\partial \underline{\mathbf{g}}^s}{\partial \theta^n} \cdot \underline{\mathbf{g}}_t + \Gamma^p_{tn}\, \underline{\mathbf{g}}^s \cdot \underline{\mathbf{g}}_p = 0 \tag{A.52b}$$

and after some algebra, we have

$$\frac{\partial \underline{\mathbf{g}}^s}{\partial \theta^n} = - \Gamma^s_{tn}\, \underline{\mathbf{g}}^t \,. \tag{A.52c}$$

Therefore,

$$\frac{\partial \underline{\mathbf{v}}}{\partial \theta^n} = \left[\frac{\partial v_p}{\partial \theta^n} - v_s\, \Gamma^s_{pn} \right] \underline{\mathbf{g}}^p \,. \tag{A.53a}$$

We now call

$$v_p|_n = \frac{\partial v_p}{\partial \theta^n} - v_s\, \Gamma^s_{pn} \,. \tag{A.53b}$$

Hence,

$$\frac{\partial \underline{\mathbf{v}}}{\partial \theta^n} = v_p|_n\, \underline{\mathbf{g}}^p \,. \tag{A.53c}$$

We call $v_p|_n$ *the covariant derivatives of the covariant components of* $\underline{\mathbf{v}}$.

We are going to show in Sect. A.7 that the $v_p|_n$ are covariant components of a second-order tensor.

A.6.2 Covariant derivatives of a general tensor

Given an arbitrary n-order tensor,

$$\mathbf{t} = t^{ij\cdots k}{}_{pq\cdots r}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \tag{A.54}$$

we can generalize the previous derivations,

$$\frac{\partial\, \mathbf{t}}{\partial \theta^n} = t^{ij\ldots k}{}_{pq\ldots r}|_n\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \tag{A.55a}$$

where

$$\begin{aligned} t^{ij\ldots k}{}_{pq\ldots r}|_n = {} & \frac{\partial t^{ij\ldots k}{}_{pq\ldots r}}{\partial \theta^n} + t^{sj\ldots k}{}_{pq\ldots r}\, \Gamma^i_{sn} + t^{is\ldots k}{}_{pq\ldots r}\, \Gamma^j_{sn} + \\ & \cdots + t^{ij\ldots s}{}_{pq\ldots r}\, \Gamma^k_{sn} - t^{ij\ldots k}{}_{sq\ldots r}\, \Gamma^s_{pn} - t^{ij\ldots k}{}_{ps\ldots r}\, \Gamma^s_{qn} \\ & - \cdots - t^{ij\ldots k}{}_{pq\ldots s}\, \Gamma^s_{rn} \end{aligned} \tag{A.55b}$$

is the *covariant derivative of the mixed components of the tensor* t.

We are going to show in Sect. A.7 that the $t^{ij\cdots k}{}_{pq\cdots r}|_n$ are mixed components of a $(n+1)$-order tensor.

Example A.11. ◀◀◀◀◀

Using Eq.(A.55b), we get

$$g_{ij}|_m = \frac{\partial g_{ij}}{\partial \theta^m} - g_{pj}\, \Gamma^p_{im} - g_{ip}\, \Gamma^p_{jm}$$

and taking into account Example A.9, we get

$$g_{ij}|_m = 0 \quad .$$

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A.7 Gradient of a tensor

Let t be a general n-order tensor,

$$\mathbf{t} = t^{ij\ldots k}{}_{pq\ldots r}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \,. \tag{A.56}$$

We define the *gradient of the tensor* t as:

$$\underline{\boldsymbol{\nabla}}\, \mathbf{t} = \underline{\mathbf{g}}^n \frac{\partial}{\partial \theta^n} \left[t^{ij\ldots k}{}_{pq\ldots r}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \right] . \tag{A.57}$$

Using the quotient rule and taking into account that due to the *definition of gradient*,

$$d\,\mathbf{t} = d\underline{\mathbf{r}} \cdot \underline{\boldsymbol{\nabla}}\, \mathbf{t} \tag{A.58}$$

and that $d\mathbf{t}$ is an n-order tensor while $d\underline{\mathbf{r}} = d\theta^n\, \underline{\mathbf{g}}_n$ is a vector, we conclude that $\underline{\boldsymbol{\nabla}}\, \mathbf{t}$ is a $(n+1)$-order tensor.

Using Eq. (A.55a), we can rewrite Eq. (A.57) as:

$$\underline{\boldsymbol{\nabla}}\, \mathbf{t} = t^{ij\ldots k}{}_{pq\ldots r}|_n\, \underline{\mathbf{g}}^n\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \,. \tag{A.59}$$

Therefore, the $t^{ij\ldots k}{}_{pq\ldots r}|_n$ are mixed components of a $(n+1)$-order tensor.

In the particular case of $\mathbf{t}$ being a vector, it is now evident that $v^p|_n$ are mixed components and the $v_p|_n$ are covariant components of the second-order tensor, then

$$\underline{\boldsymbol{\nabla}\mathbf{v}} = v^p|_n\, \underline{\mathbf{g}}^n\, \underline{\mathbf{g}}_p = v_p|_n\, \underline{\mathbf{g}}^n\, \underline{\mathbf{g}}^p \,. \tag{A.60}$$

Example A.12. ◀◀◀◀◀

We are going to show that if the components of a given tensor $\mathbf{t}$ are constant in a Cartesian system, then in any curvilinear coordinate system in the Euclidean space, the covariant derivatives of the components of $\mathbf{t}$ are zero. In a Cartesian system $\{ \hat{z}^\alpha \}$, using Eq. (A.59), we get

$$\underline{\boldsymbol{\nabla}}\ \mathbf{t} = \frac{\partial \hat{t}^{\alpha\beta\ldots\gamma}{}_{\pi\sigma\ldots\tau}}{\partial \hat{z}^\theta}\ \underline{\mathbf{e}}^\theta\ \underline{\mathbf{e}}_\alpha\ \underline{\mathbf{e}}_\beta\ \cdots\ \underline{\mathbf{e}}_\gamma\ \underline{\mathbf{e}}^\pi\ \underline{\mathbf{e}}^\sigma\ \cdots\ \underline{\mathbf{e}}^\tau$$

($\underline{\mathbf{e}}^\alpha = \underline{\mathbf{e}}_\alpha$ in a Cartesian system).

If the Cartesian components of t are constant,

$$\frac{\partial \hat{t}^{\alpha\beta\ldots\gamma}{}_{\pi\sigma\ldots\tau}}{\partial \hat{z}^\theta} = 0 \ .$$

Hence, we get

$$\underline{\boldsymbol{\nabla}}\ \mathbf{t} = \underline{\underline{\mathbf{0}}} \quad .$$

Since the above is a tensorial equation, it has to be fulfilled in any coordinate system. In particular, in a system $\{ \theta^i \}$

$$t^{ij\ldots k}{}_{pq\ldots r}|_n = 0 \ .$$

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A.8 Divergence of a tensor

Let $\mathbf{t}$ be a general n-order tensor,

$$\mathbf{t} = t^{ij\ldots k}{}_{pq\ldots r}\ \underline{\mathbf{g}}_i\ \underline{\mathbf{g}}_j\ \cdots\ \underline{\mathbf{g}}_k\ \underline{\mathbf{g}}^p\ \underline{\mathbf{g}}^q\ \cdots\ \underline{\mathbf{g}}^r \ , \tag{A.61}$$

we define the *divergence of the tensor* t as:

$$\underline{\boldsymbol{\nabla}} \cdot \mathbf{t} = \underline{\mathbf{g}}^n\ \frac{\partial}{\partial \theta^n} \cdot \left[t^{ij\ldots k}{}_{pq\ldots r}\ \underline{\mathbf{g}}_i\ \underline{\mathbf{g}}_j\ \cdots\ \underline{\mathbf{g}}_k\ \underline{\mathbf{g}}^p\ \underline{\mathbf{g}}^q\ \cdots\ \underline{\mathbf{g}}^r \right] . \tag{A.62a}$$

After some algebra, we get

$$\underline{\boldsymbol{\nabla}} \cdot \mathbf{t} = t^{ij\ldots k}{}_{pq\ldots r}|_i\ \underline{\mathbf{g}}_j\ \cdots\ \underline{\mathbf{g}}_k\ \underline{\mathbf{g}}^p\ \underline{\mathbf{g}}^q\ \cdots\ \underline{\mathbf{g}}^r \ . \tag{A.62b}$$

When we write $\mathbf{t}$ as

$$\mathbf{t} = t_i{}^{j\ldots k}{}_{pq\ldots r}\ \underline{\mathbf{g}}^i\ \underline{\mathbf{g}}_j\ \cdots\ \underline{\mathbf{g}}_k\ \underline{\mathbf{g}}^p\ \underline{\mathbf{g}}^q\ \cdots\ \underline{\mathbf{g}}^r \tag{A.63a}$$

its divergence is

$$\underline{\boldsymbol{\nabla}} \cdot \mathbf{t} = g^{ni}\ t_i{}^{j\ldots k}{}_{pq\ldots r}|_n\ \underline{\mathbf{g}}_j\ \cdots\ \underline{\mathbf{g}}_k\ \underline{\mathbf{g}}^p\ \underline{\mathbf{g}}^q\ \cdots\ \underline{\mathbf{g}}^r \ . \tag{A.63b}$$

The divergence of an n-order tensor is a $(n-1)$-order tensor.

In the particular case of a vector,

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{v}} = v^n|_n = g^{ni}\ v_i|_n \ , \tag{A.64}$$

the divergence of a vector is a scalar.

A.9 Laplacian of a tensor

Let $\mathbf{t}$ be a general n-order tensor,

$$\mathbf{t} = t^{ij\ldots k}{}_{pq\ldots r}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \tag{A.65}$$

we define the *Laplacian of the tensor* $\mathbf{t}$ as

$$\nabla^2\, \mathbf{t} = \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{\nabla}}\, \mathbf{t}\ . \tag{A.66}$$

Using Eqs. (A.59) and (A.62a-A.62b) and after lengthy algebra, we obtain

$$\nabla^2\, \mathbf{t} = t^{ij\ldots k}{}_{pq\ldots r}|_{nl}\; g^{nl}\, \underline{\mathbf{g}}_i\, \underline{\mathbf{g}}_j \cdots \underline{\mathbf{g}}_k\, \underline{\mathbf{g}}^p\, \underline{\mathbf{g}}^q \cdots \underline{\mathbf{g}}^r \tag{A.67a}$$

where

$$\begin{aligned}
t^{ij\ldots k}{}_{pq\ldots r}|_{nl} \;=\;& \frac{\partial^2 t^{ij\ldots k}{}_{pq\ldots r}}{\partial\theta^n\, \partial\theta^l} + \frac{\partial t^{sj\ldots k}{}_{pq\ldots r}}{\partial\theta^l}\, \Gamma^i_{sn} + t^{sj\ldots k}{}_{pq\ldots r}\, \frac{\partial \Gamma^i_{sn}}{\partial\theta^l} \\
&+ \frac{\partial t^{is\ldots k}{}_{pq\ldots r}}{\partial\theta^l}\, \Gamma^j_{sn} + t^{is\ldots k}{}_{pq\ldots r}\, \frac{\partial \Gamma^j_{sn}}{\partial\theta^l} + \cdots \qquad\qquad \text{(A.67b)}\\
&+ \frac{\partial t^{ij\ldots s}{}_{pq\ldots r}}{\partial\theta^l}\, \Gamma^k_{sn} + t^{ij\ldots s}{}_{pq\ldots r}\, \frac{\partial \Gamma^k_{sn}}{\partial\theta^l} - \frac{\partial t^{ij\ldots k}{}_{sq\ldots r}}{\partial\theta^l}\, \Gamma^s_{pn} \\
&- t^{ij\ldots k}{}_{sq\ldots r}\, \frac{\partial \Gamma^s_{pn}}{\partial\theta^l} - \frac{\partial t^{ij\ldots k}{}_{ps\ldots r}}{\partial\theta^l}\, \Gamma^s_{qn} - t^{ij\ldots k}{}_{ps\ldots r}\, \frac{\partial \Gamma^s_{qn}}{\partial\theta^l} \\
&- \cdots - \frac{\partial t^{ij\ldots k}{}_{pq\ldots s}}{\partial\theta^l}\, \Gamma^s_{rn} - t^{ij\ldots k}{}_{pq\ldots s}\, \frac{\partial \Gamma^s_{rn}}{\partial\theta^l} - t^{ij\ldots k}{}_{pq\ldots r}|_s\, \Gamma^s_{nl} \\
&+ t^{sj\ldots k}{}_{pq\ldots r}|_n\, \Gamma^i_{sl} + t^{is\ldots k}{}_{pq\ldots r}|_n\, \Gamma^j_{sl} + \cdots + t^{ij\ldots s}{}_{pq\ldots r}|_n\, \Gamma^k_{sl} \\
&- t^{ij\ldots k}{}_{sq\ldots r}|_n\, \Gamma^s_{pl} - t^{ij\ldots k}{}_{ps\ldots r}|_n\, \Gamma^s_{ql} - \cdots - t^{ij\ldots k}{}_{pq\ldots s}|_n\, \Gamma^s_{rl}.
\end{aligned}$$

The Laplacian of a n-order tensor is another n-order tensor.

Example A.13. ◀◀◀◀◀

In the same way we proved the lemma in Example A.12 we can show that if the components of a given tensor $\mathbf{t}$ in a Cartesian $\{\ \hat{z}^\alpha\ \}$ system have zero second derivatives, i.e.

$$\frac{\partial^2 \hat{t}^{\alpha\beta\ldots\gamma}{}_{\pi\sigma\ldots\tau}}{\partial z^\nu\, \partial z^\mu} = 0$$

then in any curvilinear coordinate system $\{\theta^i\}$ in the Euclidean space, we get

$$t^{ij\ldots k}{}_{pq\ldots r}|_{nl} = 0\ .$$

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A.10 Rotor of a tensor

Let $\mathbf{t}$ be a general n-order tensor,

$$\mathbf{t} \;=\; t^{ij\ldots k}{}_{pq\ldots r}\; \underline{\mathbf{g}}_i\; \underline{\mathbf{g}}_j\; \cdots\; \underline{\mathbf{g}}_k\; \underline{\mathbf{g}}^p\; \underline{\mathbf{g}}^q\; \cdots\; \underline{\mathbf{g}}^r, \tag{A.68}$$

we define the *rotor of the tensor* $\mathbf{t}$ as:

$$\begin{aligned} \underline{\boldsymbol{\nabla}} \times \mathbf{t} &= \underline{\mathbf{g}}^n \frac{\partial}{\partial \theta^n} \times \left[t^{ij\ldots k}{}_{pq\ldots r}\; \underline{\mathbf{g}}_i\; \underline{\mathbf{g}}_j\; \cdots\; \underline{\mathbf{g}}_k\; \underline{\mathbf{g}}^p\; \underline{\mathbf{g}}^q\; \cdots\; \underline{\mathbf{g}}^r \right] \\ &= \underline{\mathbf{g}}^n \times t^{ij\ldots k}{}_{pq\ldots r}|_n\; \underline{\mathbf{g}}_i\; \underline{\mathbf{g}}_j\; \cdots\; \underline{\mathbf{g}}_k\; \underline{\mathbf{g}}^p\; \underline{\mathbf{g}}^q\; \cdots\; \underline{\mathbf{g}}^r\;. \end{aligned} \tag{A.69a}$$

Using Eq. (A.41),

$$\underline{\boldsymbol{\nabla}} \times \mathbf{t} \;=\; \varepsilon_{nim}\; t^{ij\cdot k}{}_{pq\cdot r}|_l\; g^{nl}\; \underline{\mathbf{g}}^m\; \underline{\mathbf{g}}_j\; \cdots\; \underline{\mathbf{g}}_k\; \underline{\mathbf{g}}^p\; \underline{\mathbf{g}}^q\; \cdots\; \underline{\mathbf{g}}^r\;. \tag{A.69b}$$

The rotor of a $n-$order tensor is another $n-$order tensor.
In the particular case of a vector

$$\underline{\boldsymbol{\nabla}} \times \underline{\mathbf{v}} \;=\; \varepsilon_{ijk}\; v^j|_n\; g^{ni}\; \underline{\mathbf{g}}^k \;=\; \varepsilon^{ijk}\; v_j|_i\; \underline{\mathbf{g}}_k, \tag{A.70}$$

the rotor of a vector is a vector.

A.11 The Riemann-Christoffel tensor

Using Eqs. (A.67a-A.67b) to calculate the Laplacian of an arbitrary vector $\underline{\mathbf{v}}$, we obtain:

$$\nabla^2 \underline{\mathbf{v}} \;=\; v^i|_{nl}\; g^{nl}\; \underline{\mathbf{g}}_i \tag{A.71a}$$

where

$$\begin{aligned} v^i|_{nl} \;=\; & \frac{\partial^2 v^i}{\partial \theta^n\, \partial \theta^l} + \frac{\partial v^s}{\partial \theta^l}\, \Gamma^i_{sn} - \frac{\partial v^i}{\partial \theta^s}\, \Gamma^s_{nl} \\ & + \frac{\partial v^s}{\partial \theta^n}\, \Gamma^i_{sl} + \left[\frac{\partial \Gamma^i_{sn}}{\partial \theta^l} - \Gamma^i_{sp}\, \Gamma^p_{nl} + \Gamma^p_{sn}\, \Gamma^i_{pl} \right] v^s\;. \end{aligned} \tag{A.71b}$$

Using the quotient rule it is easy to show that the $v^i|_{nl}$ are the mixed components of a third-order tensor.

Since we are working in the *Euclidean space* where $\Gamma^s_{nl} \;=\; \Gamma^s_{ln}$ (Eq.(A.49)), we write

$$v^i|_{nl} - v^i|_{ln} \;=\; \left[\frac{\partial \Gamma^i_{sn}}{\partial \theta^l} - \frac{\partial \Gamma^i_{sl}}{\partial \theta^n} + \Gamma^t_{sn}\, \Gamma^i_{tl} - \Gamma^t_{sl}\, \Gamma^i_{tn} \right] v^s\;. \tag{A.72a}$$

Using again the quotient rule, we realize that the term between brackets on the r.h.s. of the above equation contains the mixed components (one

contravariant index and three covariant ones) of a fourth-order tensor: the *Riemmann-Christoffel tensor* ($\underline{\underline{\underline{\underline{\mathbf{R}}}}}$). Hence,

$$v^i|_{nl} \; - \; v^i|_{ln} \; = \; R_s{}^i{}_{ln} \, v^s \; . \tag{A.72b}$$

In any *Cartesian system,* we have

$$v^\alpha|_{\nu\mu} \; = \; \frac{\partial^2 v^\alpha}{\partial z^\nu \; \partial z^\mu} \tag{A.73a}$$

$$v^\alpha|_{\mu\nu} \; = \; \frac{\partial^2 v^\alpha}{\partial z^\mu \; \partial z^\nu} \; = \; v^\alpha|_{\nu\mu} \tag{A.73b}$$

and therefore, using the result in Example A.13, in any curvilinear system in the *Euclidean space*, we have

$$v^i|_{nl} \; - \; v^i|_{ln} \; = \; 0 \; . \tag{A.73c}$$

Therefore,

$$R_s{}^i{}_{ln} \; = \; 0 \; , \tag{A.73d}$$

that is to say, in the *Euclidean space*,

$$\underline{\underline{\underline{\underline{\mathbf{R}}}}} \; = \; \underline{\underline{\underline{\underline{\mathbf{0}}}}} \; . \tag{A.74}$$

In the *Euclidean space,* we can also prove that the following relation holds

$$v_i|_{nl} \; - \; v_i|_{ln} \; = \; R^s{}_{inl} \; v_s \tag{A.75a}$$

where

$$R^s{}_{inl} \; = \; \frac{\partial \Gamma^s_{il}}{\partial \theta^n} \; - \; \frac{\partial \Gamma^s_{in}}{\partial \theta^l} \; + \; \Gamma^s_{tn} \, \Gamma^t_{il} \; - \; \Gamma^s_{tl} \, \Gamma^t_{in} \; . \tag{A.75b}$$

We can use the metric tensor components to lower the contravariant index; hence,

$$R_{ijkl} \; = \; g_{si} \; R^s{}_{jkl} \; . \tag{A.76a}$$

Therefore,

$$R_{ijkl} \; = \; g_{si} \; \left[\; \frac{\partial \Gamma^s_{jl}}{\partial \theta^k} \; - \; \frac{\partial \Gamma^s_{jk}}{\partial \theta^l} \; + \; \Gamma^s_{tk} \, \Gamma^t_{jl} \; - \; \Gamma^s_{tl} \, \Gamma^t_{jk} \; \right] \; . \tag{A.76b}$$

We now define the *Christoffel symbol of the first kind*, Γ_{ijk}, as:

$$\Gamma_{ijk} \; = \; g_{sk} \; \Gamma^s_{ij} \tag{A.77a}$$

$$\Gamma^s_{ij} \; = \; g^{sk} \; \Gamma_{ijk} \tag{A.77b}$$

using the above in Eq. (A.76b) we get,

$$R_{ijkl} = \frac{\partial \mathit{\Gamma}_{jli}}{\partial \theta^k} - \frac{\partial \mathit{\Gamma}_{jki}}{\partial \theta^l} + \mathit{\Gamma}^s_{jl} \left[\mathit{\Gamma}_{ski} - \frac{\partial g_{si}}{\partial \theta^k} \right] + \mathit{\Gamma}^s_{jk} \left[\frac{\partial g_{si}}{\partial \theta^l} - \mathit{\Gamma}_{sli} \right] . \tag{A.77c}$$

It is very important to realize that Eqs.(A.77a) and (A.77b) are not standard operations to go from contravariant tensorial components to covariant tensorial components and vice versa because *we have already established that the Christoffel symbols are not tensorial components.*

The result in Example A.9 can now be rewritten as:

$$\frac{\partial g_{si}}{\partial \theta^l} = \mathit{\Gamma}_{sli} + \mathit{\Gamma}_{ils} . \tag{A.77d}$$

Using the above in Eq. (A.77c) and taking into account that in the Euclidean space $\mathit{\Gamma}_{ils} = \mathit{\Gamma}_{lis}$, we get

$$R_{ijkl} = \frac{\partial \mathit{\Gamma}_{jli}}{\partial \theta^k} - \frac{\partial \mathit{\Gamma}_{jki}}{\partial \theta^l} - \mathit{\Gamma}^s_{jl}\, \mathit{\Gamma}_{kis} + \mathit{\Gamma}^s_{jk}\, \mathit{\Gamma}_{lis} . \tag{A.77e}$$

In what follows, we will prove the identities:

$$(i) \quad R_{ijkl} = - R_{ijlk} , \tag{A.78a}$$

$$(ii) \quad R_{ijkl} = - R_{jikl} , \tag{A.78b}$$

$$(iii) \quad R_{ijkl} = R_{klij} . \tag{A.78c}$$

(i) $R_{ijkl} = - R_{ijlk}$

Using Eq. (A.77e), we write

$$\begin{aligned} R_{ijkl} &= \frac{\partial \mathit{\Gamma}_{jli}}{\partial \theta^k} - \frac{\partial \mathit{\Gamma}_{jki}}{\partial \theta^l} - \mathit{\Gamma}^s_{jl}\, \mathit{\Gamma}_{kis} + \mathit{\Gamma}^s_{jk}\, \mathit{\Gamma}_{lis} \\ &= - \left[\frac{\partial \mathit{\Gamma}_{jki}}{\partial \theta^l} - \frac{\partial \mathit{\Gamma}_{jli}}{\partial \theta^k} - \mathit{\Gamma}^s_{jk}\, \mathit{\Gamma}_{lis} + \mathit{\Gamma}^s_{jl}\, \mathit{\Gamma}_{kis} \right] \\ &= - R_{ijlk} . \end{aligned} \tag{A.79a}$$

(ii) $R_{ijkl} = - R_{jikl}$

Since we are working in the Euclidean space, $\mathit{\Gamma}^s_{ij} = \mathit{\Gamma}^s_{ji}$ and $\mathit{\Gamma}_{ijk} = \mathit{\Gamma}_{jik}$. Also, we can rewrite the result of Example A.10 as:

$$\mathit{\Gamma}_{abc} = \frac{1}{2} \left(\frac{\partial g_{ac}}{\partial \theta^b} + \frac{\partial g_{bc}}{\partial \theta^a} - \frac{\partial g_{ab}}{\partial \theta^c} \right) . \tag{A.79b}$$

Using the above in Eq.(A.77e), we obtain, after some algebra:

$$R_{ijkl} = \frac{1}{2} \left[\frac{\partial^2 g_{li}}{\partial\theta^j \, \partial\theta^k} + \frac{\partial^2 g_{jk}}{\partial\theta^i \, \partial\theta^l} - \frac{\partial^2 g_{jl}}{\partial\theta^i \, \partial\theta^k} - \frac{\partial^2 g_{ki}}{\partial\theta^j \, \partial\theta^l} \right]$$
$$+ \, g^{sa} \left[\mathit{\Gamma}_{jka} \, \mathit{\Gamma}_{lis} - \mathit{\Gamma}_{jla} \, \mathit{\Gamma}_{kis} \right] . \qquad \text{(A.79c)}$$

Changing the order of the indices, we obtain

$$\begin{aligned} R_{jikl} &= \frac{1}{2} \left[\frac{\partial^2 g_{lj}}{\partial\theta^i \, \partial\theta^k} + \frac{\partial^2 g_{ik}}{\partial\theta^j \, \partial\theta^l} - \frac{\partial^2 g_{il}}{\partial\theta^j \, \partial\theta^k} - \frac{\partial^2 g_{kj}}{\partial\theta^i \, \partial\theta^l} \right] \\ &\quad + g^{sa} \left[\mathit{\Gamma}_{ika} \, \mathit{\Gamma}_{ljs} - \mathit{\Gamma}_{ila} \, \mathit{\Gamma}_{kjs} \right] \\ &= -\frac{1}{2} \left[\frac{\partial^2 g_{li}}{\partial\theta^j \, \partial\theta^k} + \frac{\partial^2 g_{jk}}{\partial\theta^i \, \partial\theta^l} - \frac{\partial^2 g_{jl}}{\partial\theta^i \, \partial\theta^k} - \frac{\partial^2 g_{ki}}{\partial\theta^j \, \partial\theta^l} \right] \\ &\quad - g^{sa} \left[\mathit{\Gamma}_{lia} \, \mathit{\Gamma}_{jks} - \mathit{\Gamma}_{kia} \, \mathit{\Gamma}_{jls} \right] \\ &= - R_{ijkl} \, . \end{aligned} \qquad \text{(A.79d)}$$

(iii) $R_{ijkl} = R_{klij}$
Using Eq. (A.79c), we can write

$$\begin{aligned} R_{klij} &= \frac{1}{2} \left[\frac{\partial^2 g_{jk}}{\partial\theta^l \, \partial\theta^i} + \frac{\partial^2 g_{li}}{\partial\theta^k \, \partial\theta^j} - \frac{\partial^2 g_{lj}}{\partial\theta^k \, \partial\theta^i} - \frac{\partial^2 g_{ik}}{\partial\theta^l \, \partial\theta^j} \right] \\ &\quad + g^{sa} \left[\mathit{\Gamma}_{lia} \, \mathit{\Gamma}_{jks} - \mathit{\Gamma}_{lja} \, \mathit{\Gamma}_{iks} \right] \\ &= \frac{1}{2} \left[\frac{\partial^2 g_{li}}{\partial\theta^j \, \partial\theta^k} + \frac{\partial^2 g_{jk}}{\partial\theta^i \, \partial\theta^l} - \frac{\partial^2 g_{jl}}{\partial\theta^i \, \partial\theta^k} - \frac{\partial^2 g_{ki}}{\partial\theta^j \, \partial\theta^l} \right] \\ &\quad + g^{sa} \left[\mathit{\Gamma}_{jks} \, \mathit{\Gamma}_{lia} - \mathit{\Gamma}_{jla} \, \mathit{\Gamma}_{kis} \right] \\ &= R_{ijkl} \, . \end{aligned} \qquad \text{(A.79e)}$$

A.12 The Bianchi identity

A second-order tensor $\underline{\underline{\mathbf{g}}}$ can be considered a metric tensor in a *Euclidean space* if it fulfills the set of equations $R_{ijkl} = 0$, derived from Eq.(A.74). However, between those equations, certain relations exist that we are going to demonstrate in this Section.

Using Eq. (A.55b), we can write

$$R^i{}_{jkl}|_m = \frac{\partial R^i{}_{jkl}}{\partial\theta^m} + R^s{}_{jkl} \, \mathit{\Gamma}^i_{sm} - R^i{}_{skl} \, \mathit{\Gamma}^s_{jm}$$
$$- R^i{}_{jsl} \, \mathit{\Gamma}^s_{km} - R^i{}_{jks} \, \mathit{\Gamma}^s_{lm} \, , \qquad \text{(A.80a)}$$

and with the help of Eq. (A.75b), we get, *in the Euclidean space*,

$$
\begin{aligned}
R^{i}{}_{jkl}|_m \;=\;& \frac{\partial^2 \Gamma^i_{jl}}{\partial\theta^k\,\partial\theta^m} - \frac{\partial^2 \Gamma^i_{jk}}{\partial\theta^l\,\partial\theta^m} + \frac{\partial \Gamma^i_{sk}}{\partial\theta^m}\,\Gamma^s_{jl} + \Gamma^i_{sk}\,\frac{\partial \Gamma^s_{jl}}{\partial\theta^m} \\
& - \frac{\partial\;^{i}}{\Gamma\;_{sl}}\partial\theta^m\,\Gamma^s_{jk} - \Gamma^i_{sl}\,\frac{\partial \Gamma^s_{jk}}{\partial\theta^m} + \frac{\partial \Gamma^s_{jl}}{\partial\theta^k}\,\Gamma^i_{sm} - \frac{\partial \Gamma^s_{jk}}{\partial\theta^l}\,\Gamma^i_{sm} \\
& + \Gamma^s_{tk}\,\Gamma^t_{jl}\,\Gamma^i_{sm} - \Gamma^s_{tl}\,\Gamma^t_{jk}\,\Gamma^i_{sm} - \frac{\partial \Gamma^i_{sl}}{\partial\theta^k}\,\Gamma^s_{jm} + \frac{\partial \Gamma^i_{sk}}{\partial\theta^l}\,\Gamma^s_{jm} \\
& - \Gamma^i_{tk}\,\Gamma^t_{sl}\,\Gamma^s_{jm} + \Gamma^i_{tl}\,\Gamma^t_{sk}\,\Gamma^s_{jm} - \frac{\partial \Gamma^i_{jl}}{\partial\theta^s}\,\Gamma^s_{km} + \frac{\partial \Gamma^i_{js}}{\partial\theta^l}\,\Gamma^s_{km} \\
& - \Gamma^i_{ts}\,\Gamma^t_{jl}\,\Gamma^s_{km} + \Gamma^i_{tl}\,\Gamma^t_{js}\,\Gamma^s_{km} - \frac{\partial \Gamma^i_{js}}{\partial\theta^k}\,\Gamma^s_{lm} + \frac{\partial \Gamma^i_{jk}}{\partial\theta^s}\,\Gamma^s_{lm} \\
& - \Gamma^i_{tk}\,\Gamma^t_{js}\,\Gamma^s_{lm} + \Gamma^i_{ts}\,\Gamma^t_{jk}\,\Gamma^s_{lm} \;. \qquad \text{(A.80b)}
\end{aligned}
$$

We can develop similar expressions for $R^{i}{}_{jlm}|_k$ and $R^{i}{}_{jmk}|_l$ and remembering that in the Euclidean space $\Gamma^a_{bc} = \Gamma^a_{cb}$, we finally obtain the *Bianchi identity*:

$$R^{i}{}_{jkl}|_m + R^{i}{}_{jlm}|_k + R^{i}{}_{jmk}|_l = 0 \;. \qquad \text{(A.81)}$$

Starting from,

$$R_{ijkl} = g_{pi}\, R^{p}{}_{jkl} \qquad \text{(A.82a)}$$

and using the result in Example A.11, we have

$$R_{ijkl}|_m = g_{pi}\, R^{p}{}_{jkl}|_m \;. \qquad \text{(A.82b)}$$

Hence,

$$R_{ijkl}|_m + R_{ijlm}|_k + R_{ijmk}|_l = g_{pi}\left(R^{p}{}_{jkl}|_m + R^{p}{}_{jlm}|_k + R^{p}{}_{jmk}|_l \right) \qquad \text{(A.82c)}$$

and using Eq.(A.81), we get

$$R_{ijkl}|_m + R_{ijlm}|_k + R_{ijmk}|_l = 0 \;. \qquad \text{(A.82d)}$$

It is worth noting that the Bianchi identities are not restricted to Euclidean spaces and can be demonstrated in other spaces in which the second Christoffel symbol is also symmetric (e.g. Riemmanian spaces (McConnell 1957)).

A.13 Physical components

In an arbitrary curvilinear system $\{\,\theta^i\,\}$, we can write the n-order tensor $\mathbf{t}$, using its contravariant components and the covariant base vectors,

$$\mathbf{t} = t^{ij\ldots k}\,\underline{\mathbf{g}}_i\,\underline{\mathbf{g}}_j\,\cdots\,\underline{\mathbf{g}}_k \;. \qquad \text{(A.83)}$$

In general, the covariant base vectors:
(i) Do not have a unitary modulus.
(ii) Are not dimensionally homogeneous.

Example A.14. ◀◀◀◀◀
In a cylindrical coordinate system where θ^1 is the radius, θ^2 the polar angle and $\theta^3 \equiv z^3$,we can write

$$\begin{aligned} \underline{\mathbf{g}}_1 &= \cos\theta^2\ \underline{\mathbf{e}}_1 + \sin\theta^2\ \underline{\mathbf{e}}_2 \\ \underline{\mathbf{g}}_2 &= -\theta^1\ \sin\theta^2\ \underline{\mathbf{e}}_1 + \theta^1\ \cos\theta^2\ \underline{\mathbf{e}}_2 \\ \underline{\mathbf{g}}_3 &= \underline{\mathbf{e}}_3\ . \end{aligned}$$

Therefore,

$$\begin{aligned} \left|\underline{\mathbf{g}}_1\right| &= 1 \\ \left|\underline{\mathbf{g}}_2\right| &= \theta^1 \\ \left|\underline{\mathbf{g}}_3\right| &= 1 \end{aligned}$$

which are obviously not dimensionally homogeneous. ◀◀◀◀◀

We can rewrite Eq. (A.83) as:

$$\mathbf{t} = \sum_{i=1}^{3} \sum_{j=1}^{3} \cdots \sum_{k=1}^{3} t^{ij\ldots k} \sqrt{g_{ii}}\,\sqrt{g_{jj}} \cdots \sqrt{g_{kk}}\ \frac{\underline{\mathbf{g}}_i}{\sqrt{g_{ii}}}\ \frac{\underline{\mathbf{g}}_j}{\sqrt{g_{jj}}} \cdots \frac{\underline{\mathbf{g}}_k}{\sqrt{g_{kk}}}\ . \tag{A.84a}$$

In the above equation, we did not use the summation convention to avoid misinterpretations.

Obviously,

$$\left| \frac{\underline{\mathbf{g}}_i}{\sqrt{g_{ii}}} \right| = 1\ , \tag{A.84b}$$

and therefore the terms

$$t^{<ij\ldots k>} = t^{ij\ldots k} \sqrt{g_{ii}}\,\sqrt{g_{jj}} \cdots \sqrt{g_{kk}} \qquad (\textit{ no addition on } i, j \ldots k) \tag{A.84c}$$

are the projections of the tensor t on base vectors of unitary modulus. The terms $t^{<ij\ldots k>}$ are known as the *physical components* of the tensor t.

It is obvious that the above-defined physical components are not tensorial components and therefore, when the coordinate system is changed the physical components cannot be transformed using either a covariant or a contravariant transformation rule.

B

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